Numerical Solution of Boundary Value Problems

Weighted Residual Methods

ENME 602 Spring 2007
Dr. Eng. Mohammad Tawfik

Objectives

- In this section we will be introduced to the general classification of approximate methods
- Special attention will be paid for the weighted residual method
- Derivation of a system of linear equations to approximate the solution of an ODE will be presented using different techniques

Classification of Approximate Solutions of D.E.’s

- Discrete Coordinate Method
  - Finite difference Methods
  - Stepwise integration methods
    - Euler method
    - Runge-Kutta methods
    - Etc…
- Distributed Coordinate Method

Distributed Coordinate Methods

- Weighted Residual Methods
  - Interior Residual
    - Collocation
    - Galrekin
    - Finite Element
  - Boundary Residual
    - Boundary Element Method
- Stationary Functional Methods
  - Reyligh-Ritz methods
  - Finite Element method

Basic Concepts

- A linear differential equation may be written in the form:

$$L(f(x)) = g(x)$$

- Where *L(.) is a linear differential operator.*
- *An approximate solution maybe of the form:*

$$f(x) = \sum_{i=1}^{n} a_i \psi_i(x)$$

Basic Concepts

- Applying the differential operator on the approximate solution, you get:

$$L(f(x)) - g(x) = L\left(\sum_{i=1}^{n} a_i \psi_i(x)\right) - g(x)$$

$$= \sum_{i=1}^{n} a_i L(\psi_i(x)) - g(x) \neq 0$$

$$\sum_{i=1}^{n} a_i L(\psi_i(x)) - g(x) = R(x)$$

Residue

Handling the Residue

- The weighted residual methods are all based on minimizing the value of the residue.
- Since the residue can not be zero over the whole domain, different techniques were introduced.

General Weighted Residual Method

Objective of WRM

- As any other numerical method, the objective is to obtain of algebraic equations, that, when solved, produce a result with an acceptable accuracy.
- If we are seeking the values of a_i that would reduce the Residue ($R(x)$) allover the domain, we may integrate the residue over the domain and evaluate it!

Evaluating the Residue

$$\sum_{i=1}^{n} a_i L(\psi_i(x)) - g(x) = R(x)$$

$$a_1 L(\psi_1(x)) + a_2 L(\psi_2(x)) + \dots + a_n L(\psi_n(x)) - g(x) = R(x)$$

n unknown variables

One equation!!!

$$\int_{Domain} R(x)dx = \int_{Domain} \left(\sum_{i=1}^{n} a_i L(\psi_i(x)) - g(x) \right) dx = 0$$

Using Weighting Functions

$$\int_{Domain} w_j(x)R(x)dx = \int_{Domain} w_j(x)\left(\sum_{i=1}^{n} a_i L(\psi_i(x)) - g(x)\right)dx = 0$$

- If you can select n different weighting functions, you will produce n equations!
- You will end up with n equations in n variables.

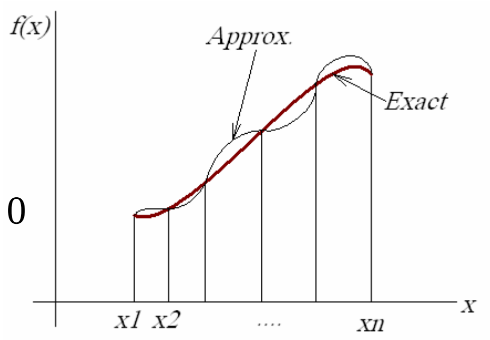
Collocation Method

- The idea behind the collocation method is similar to that behind the buttons of your shirt!
- Assume a solution, then force the residue to be zero at the *collocation* points

Collocation Method

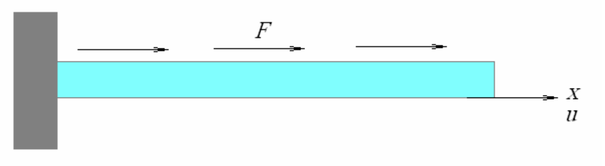
$$R(x_j) = 0$$

$$R(x_j) = \sum_{i=1}^{n} a_i L(\psi_i(x_j)) - F(x_j) = 0$$



Example Problem

The bar tensile problem



$$EA\frac{\partial^2 u}{\partial x^2}+F(x)=0$$

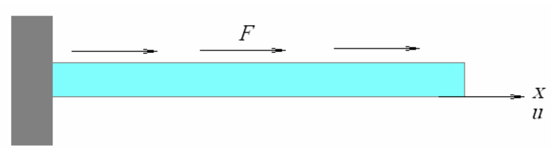
$$BC's$$

$$x=0 \Rightarrow u=0$$

$$x=l \Rightarrow du/dx=0$$

Bar application

$$EA\frac{\partial^2 u}{\partial x^2}+F(x)=0$$



$$u(x)=\sum_{i=1}^{n} a_i\psi_i(x)$$

$$EA\sum_{i=1}^{n} a_i\frac{d^2\psi_i(x)}{dx^2}+F(x)=R(x)$$

Applying the collocation method

$$EA\sum_{i=1}^{n} a_i\frac{d^2\psi_i(x_j)}{dx^2}+F(x_j)=0$$

In Matrix Form

$$\begin{bmatrix} k_{11} & k_{21} & \dots & k_{n1} \\ k_{12} & k_{22} & \dots & k_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ k_{1n} & k_{2n} & \dots & k_{nn} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{Bmatrix} = - \begin{Bmatrix} F(x_1) \\ F(x_2) \\ \vdots \\ F(x_n) \end{Bmatrix} \qquad k_{ij} = EA \left. \frac{d^2 \psi_i(x)}{dx^2} \right|_{x=x_j}$$

Solve the above system for the “generalized coordinates” a_i to get the solution for u(x)

Notes on the trial functions

- They should be at least twice differentiable!
- They should satisfy all boundary conditions!
- Those are called the “Admissibility Conditions”.

Using Admissible Functions

- For a constant forcing function, F(x)=f
- The strain at the free end of the bar should be zero (slope of displacement is zero). We may use:

$$\psi(x) = Sin\left(\frac{\pi x}{2l}\right)$$

Using the function into the DE:

$$EA\frac{d^2\psi(x)}{dx^2} = -EA\left(\frac{\pi}{2l}\right)^2 Sin\left(\frac{\pi x}{2l}\right)$$

- Since we only have one term in the series, we will select one collocation point!
- The midpoint is a reasonable choice!

$$\left[-EA\left(\frac{\pi}{2l}\right)^2 Sin\left(\frac{\pi}{4}\right)\right]\{a_1\} = -\{f\}$$

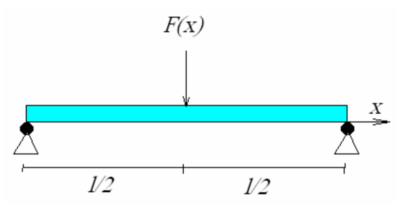
Solving:

$$a_1 = \frac{f}{EA(\pi/2l)^2 Sin(\pi/4)} = \frac{4\sqrt{2}l^2}{\pi^2}\frac{f}{EA} \approx 0.57\frac{l^2 f}{EA}$$

- Then, the approximate solution for this problem is: $u(x) \approx 0.57\frac{l^2 f}{EA} Sin\left(\frac{\pi x}{2l}\right)$
- Which gives the maximum displacement to be: $u(l) \approx 0.57\frac{l^2 f}{EA}(exact = 0.5)$
- And maximum strain to be: $u_x(0) \approx 0.9\frac{lf}{EA}(exact = 1.0)$

Homework #11

- Solve the beam bending problem, for beam displacement, for a simply supported beam with a load placed at the center of the beam using
  - Any weighting function
  - Collocation Method
- Use three term Sin series that satisfies all BC's
- Write a program that produces the results for n-term solution.



$$\frac{d^4 w}{dx^4} = F(x)$$

$$w(0) = w(l) = 0$$

$$\frac{d^2 w(0)}{dx^2} = \frac{d^2 w(l)}{dx^2} = 0$$

Exact Solution

$$
\begin{aligned}
w(x) &= \frac{x^3}{12} + \frac{13x}{60} && 0 < x < 1/2 \\
&= -\frac{x^3}{12} + \frac{x^2}{4} - \frac{7x}{15} + \frac{3}{10} && 1/2 < x < 1
\end{aligned}
$$