Προχωρημενη Μηχανικη Υλικων

Κεφάλαιο 1

Θεωρία + Ασκησεις Παπαδιά

Δυναμη P_i → μετατόπιση u_i

$$\left.\begin{aligned} u_1 &= C_{11}\cdot P_1 + C_{12}\cdot P_2 + \dots + C_{1n}\cdot P_n \\ u_2 &= C_{21}\cdot P_1 + C_{22}\cdot P_2 + \dots + C_{2n} P_n \\ &\vdots \\ u_n &= C_{n1}\cdot P_1 + C_{n2} P_2 + \dots + C_{nn}\cdot P_n \end{aligned}\right\} \boxed{u_i = \sum_{j=1}^{n} C_{ij}\cdot P_j}$$

Μια και μόνο λύση όταν

$$\begin{vmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{vmatrix} \neq 0$$

• Τελική Έκφραση Έργου : $W = \frac{1}{2}\sum_{i=1}^{n} P_i\cdot u_i$

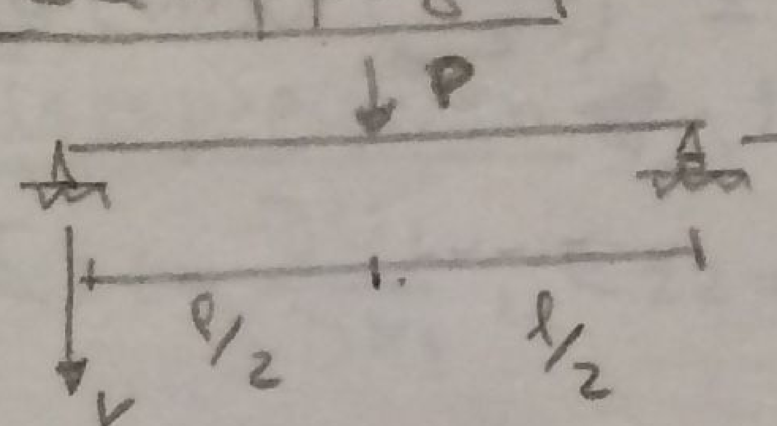
$$U = \int_0^{\ell}\left(\frac{N^2}{2EA} + \frac{M^2}{2EI} + f_s\cdot\frac{V^2}{2GA} + \frac{T^2}{2GJ}\right)dx$$

→ Μεθοδος Rayleigh - Ritz

$$u(x) = \sum_{i=1}^{n} a_i g_i(x) \quad , \quad u''(x) = \frac{d^2u}{dx^2} = \sum_{i=1}^{n} a_i\cdot g_i''(x)$$

$$\Pi = \int_0^L \frac{EI}{2}\cdot\left(\sum_{i=1}^{n} a_i\cdot g_i''(x)\right)^2 dx - \int_0^L P(x)\cdot\left(\sum_{i=1}^{n} a_i\cdot g_i\right)dx$$

Εφαρμογή



$y(x) = u(x) = a\cdot\sin\left(\frac{\pi x}{\ell}\right) \quad 0\le x\le \ell.$

Συνοριακές Συνθήκες.
$u(0) = 0 = u(\ell)$

$$U = \frac{EI}{2}\int (u''(x))^2 dx \qquad u'(x) = a\cdot\cos\left(\frac{\pi x}{\ell}\right)\cdot\frac{\pi}{\ell}$$

$$u''(x) = -a\frac{\pi^2}{\ell^2}\cdot\sin\left(\frac{\pi x}{\ell}\right) \rightarrow U = \frac{EI}{2}\int_0^{\ell}\frac{a^2\cdot\pi^4}{\ell^4}\cdot\sin^2\left(\frac{\pi x}{\ell}\right) = \frac{\pi^2 EI a^2}{4\ell^3}$$

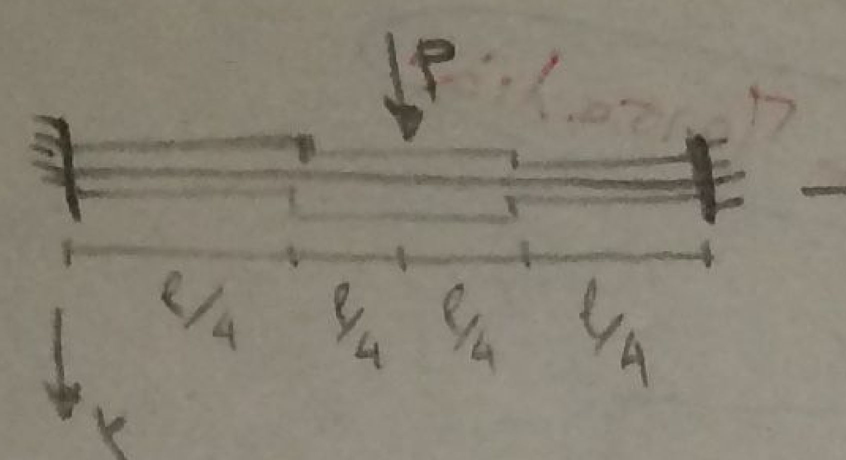
Εργο Εξωτερικών Δυνάμεων $= P\cdot u(x=\ell/2) = P\cdot a\cdot\sin\left(\pi\cdot\frac{\ell/2}{\ell}\right) = Pa\cdot\sin(\pi/2)$
$= \underline{P\cdot a}$

$$\Pi = U - Pa = \frac{\pi^4}{\ell^4}EI\cdot a^2 - Pa$$

$$\frac{\partial\Pi}{\partial a} = 0 \Leftrightarrow a = \dots$$

$\boxed{\text{Εφαρμογή}}$ ②

1) $u(x) = a_1x^2 + a_2x + a_3$



Συνοριακές Συνθήκες : $u(x=0)=0 \Leftrightarrow a_3=0$

$u(x)=a_1x^2+a_2x$, $u'(x)=2a_1x+a_2$

$u'(x=0)=0 \Leftrightarrow a_2=0$

$u(x)=a_1x^2$, $u'(x)=2a_1x$

$u(x=\ell)=0 \Leftrightarrow 2a_1\ell^2=0 \Leftrightarrow a_1=0$

2) $u(x)=a_1x^3+a_2x^2+a_3x+a_4 \quad (0\le x\le \ell/2)$

$u(x=0)=0 \Leftrightarrow a_4=0$, $u'(x=0)=0 \Leftrightarrow a_3=0$

$\boxed{u(x)=a_1x^3+a_2x^2}$, $\boxed{u'(x)=3a_1x^2+2a_2x}$

$u(x=\ell/2)=a \Leftrightarrow a \Leftrightarrow a_1=-\frac{16a}{\ell^3}$

$u'(x=\ell/2)=0 \Leftrightarrow 3a_1\cdot\frac{\ell^{\cancel{2}}}{\cancel{4}}=-2a_2\cdot\frac{\cancel{\ell}}{\cancel{2}} \Leftrightarrow 3a_1\cdot\frac{\ell}{4}=\frac{-2a_2}{2}$

$\Leftrightarrow a_2=-\frac{3\ell a_1}{4}$ (A)

$u(x)=----$, $V = 2\,\frac{EI}{2}\int_0^{\ell/4}(u''(x))^2dx + \int_{\ell/4}^{\ell/2} 2\left(\frac{2EI}{2}\right)(u''(x))^2dx$

$=---$

$\Rightarrow a_1\cdot\frac{\ell^3}{2^3}+a_2\cdot\frac{\ell^2}{2^2}=a \Leftrightarrow a_1\cdot\frac{\ell^3}{2^3}-\frac{3\ell a_1}{4}\cdot\frac{\ell^2}{2^2}=a$

$\Leftrightarrow \frac{a_1\cdot\ell^3}{2^3}\left[1-\frac{1}{2}\right]=a \Leftrightarrow a_1=-\frac{16a}{\ell^3}$ (B)

(A) (B) $\Rightarrow$

$a_2=-\frac{3\ell}{\cancel{4}}\left[-\frac{\overset{4}{\cancel{16}}a}{\ell^3}\right]=12a/\ell^2=a_2$

$\boxed{\Pi = V - Pa} \Leftrightarrow \frac{\partial\Pi}{\partial a}=0$

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2}\right) + \left(\frac{\partial^2}{\partial y^2}\right), \quad \nabla^4 = \left(\frac{\partial^4}{\partial x^4}\right) + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \left(\frac{\partial^4}{\partial y^4}\right)$$

Κεφάλαιο 2

$$T^{(1)} = \sigma_{11} \cdot e_1 + \sigma_{12} \cdot e_2 + \sigma_{13} \cdot e_3$$
$$T^{(2)} = \sigma_{21} \cdot e_1 + \sigma_{22} \cdot e_2 + \sigma_{23} \cdot e_3$$
$$T^{(3)} = \sigma_{31} \cdot e_1 + \sigma_{32} \cdot e_2 + \sigma_{33} \cdot e_3$$

σ_{ij} → stress tensor (Τανυστής Τάσεων).

Κύριες Κατευθύνσεις και Τάσεις

$$\begin{vmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma \end{vmatrix} = 0 \iff \boxed{\sigma^3 - I \cdot \sigma^2 + II\,\sigma - III = 0}$$

όπου $I = \sigma_{11} + \sigma_{22} + \sigma_{33}$, $II = \frac{1}{2}\left(\sigma_{ii} \cdot \sigma_{jj} - \sigma_{ij}\sigma_{ji}\right)$

$$III = |\sigma_{ij}|$$

Παραμόρφωση

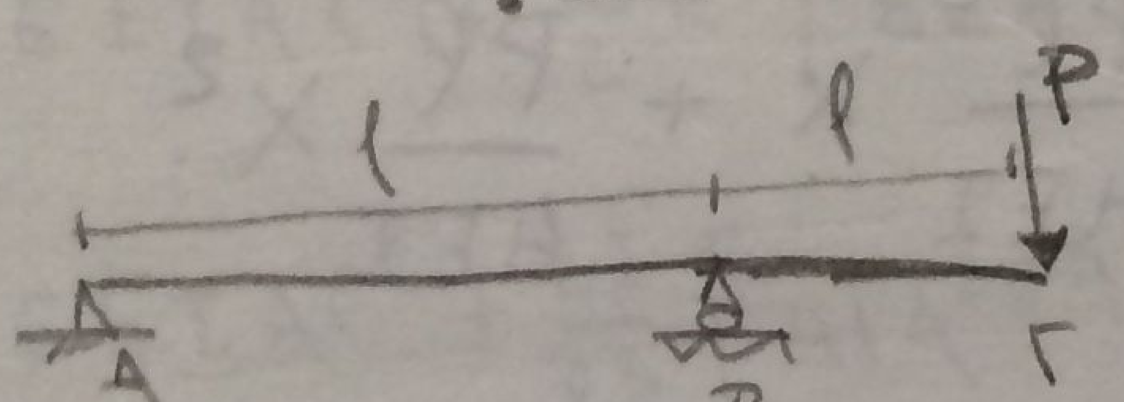
$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \epsilon_{22} = \frac{\partial u_2}{\partial x_2}, \quad \epsilon_{33} = \frac{\partial u_3}{\partial x_3}, \quad \epsilon_{12} = \epsilon_{21} = \frac{1}{2}\left[\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right]$$

$$\epsilon_{23} = \frac{1}{2}\left[\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right], \quad \epsilon_{31} = \epsilon_{13} = \frac{1}{2}\left[\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right]$$

Ασκήσεις sos

$$\Pi = U - [\text{Έργο Εξωτερικών Δυνάμεων}]$$

1



Βύθιση στο Γ = ;

Ασκήσεις Πασσαλία

1, 2, 3, 4

5, 6, 7

A: άρθρωση [οριζόντια + κατακόρυφη μετατόπιση = 0]

B: κύλιση [κατακόρυφη μετατόπιση = 0]

$$u_{(x)} = c_0 + c_1 \cdot x + c_2 \cdot x^2.$$

Συνοριακες συνθηκες: $\begin{cases} u_{(x=0)} = 0 \Leftrightarrow c_0 = 0 \\ u_{(x=\ell)} = 0 \Leftrightarrow c_1 \cdot \ell + c_2 \cdot \ell^2 = 0 \Leftrightarrow c_1 = -c_2 \cdot \ell \end{cases}$

$$u_{(x)} = -c_2 \cdot \ell x + c_2 \cdot x^2 \quad , \quad u'_{(x)} = -c_2 \cdot \ell + 2 c_2 \cdot x$$

$$u''(x) = 2c_2$$

$$U = \frac{EI}{2} \int_0^{2\ell} (u''_{(x)})^2 dx = \frac{EI}{2} \int_0^{2\ell} 4c_2^2 \, dx = \frac{EI}{2} \cdot 4c_2^2 \cdot 2\ell =$$

$$\boxed{U = 4EI\ell c_2^2}$$

Σε ολο τον φορεα SOS

$$\text{Εργο Εξωτερικων Δυναμεων} = P \cdot u_{(x=2\ell)}$$

$$= P \cdot [-2c_2 \cdot \ell^2 + 4c_2 \ell^2]$$

$$\Pi = 4EI\ell c_2^2 + 2c_2 P \ell^2 - 4c_2 P \ell^2 =$$

$$\boxed{\Pi = 4EI\ell c_2^2 - 2c_2 P \ell^2}$$

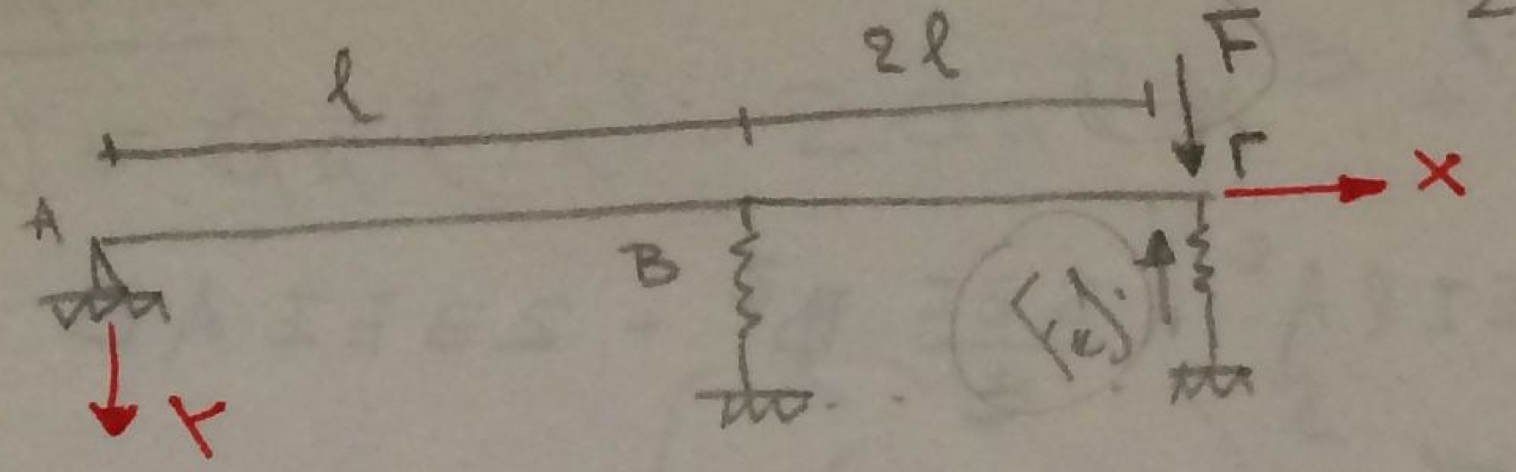
$$\frac{\partial \Pi}{\partial c_2} = 0 \Leftrightarrow 8EI\ell c_2 - 2P\ell^2 = 0 \Leftrightarrow$$

$$c_2 = \frac{2P\ell}{8EI} \Leftrightarrow \boxed{c_2 = \frac{P\ell}{4EI}}$$

$$\text{Αρα } u_{(x)} = c_1 \cdot x + \frac{P\ell}{4EI} x^2 = -\frac{P\ell^2}{4EI} x + \frac{P\ell}{4EI} x^2.$$

Σταθερά Ελατηρίου $K = \frac{EI}{L^3}$ //

2



$$\boxed{F = F_{ελ} = K \cdot U_{(x=3\ell)}}$$

A: αρθρωση $\Longrightarrow U_{(x=0)} = 0$

$U_{(x)} = Ax^2 + Bx + \Gamma$.

$U_{(x=0)} = 0 \Leftrightarrow \Gamma = 0$ Αρα $U_{(x)} = Ax^2 + Bx$

$U'_{(x)} = 2Ax + B$, $U''_{(x)} = 2A$.

$\Pi = U - [$Εργο Εξωτερικών Δυνάμεων$]$

$$V = \frac{EI}{2} \int_0^{3\ell} (2A)^2 dx = \frac{EI}{2} \cdot 4A^2 \cdot 3\ell =$$

$$\boxed{U = 6EIA^2\ell} + \frac{1}{2} K \cdot (U_{(x=\ell)})^2 + \frac{1}{2} K \cdot (U_{(x=3\ell)})^2$$

Ενεργεια Ελατηριου !

$$V = 6EIA^2\ell + \frac{1}{2}k\left[A\ell^2 + B\ell\right]^2 + \frac{1}{2}k\left[9A\ell^2 + 3B\ell\right]^2$$

$$6EIA \cdot \ell^2 + \frac{1}{2}k\left[A^2\ell^4 + B^2\ell^2 + 2AB\ell^3\right] + \frac{1}{2}k\left[81A^2\ell^4 + 9B^2\ell^2 + 54AB\ell^3\right]$$

$$= 6EIA\ell^2 + \frac{1}{2}k\left[82A^2\ell^4 + 10B^2\ell^2 + 56AB\ell^3\right]$$

$$= 6EIA\ell^2 + \frac{EI}{\ell^3}\left[41A^2\ell^4 + 5B^2\ell^2 + 28AB\ell^3\right]$$

$$6EIA\ell^2 + 41A^2EI\ell + \frac{5B^2EI}{\ell} + 28ABEI = V$$

$\Pi = U - (\text{Έργο εξ. Δυν.})$

$= \Pi(A,B) = 6EI\ell^2 A + 41EI\ell A^2 + \frac{5EI}{\ell}B^2 + 28EI\,AB$

$-9F\ell^2 A - 3F\ell B = \Pi$

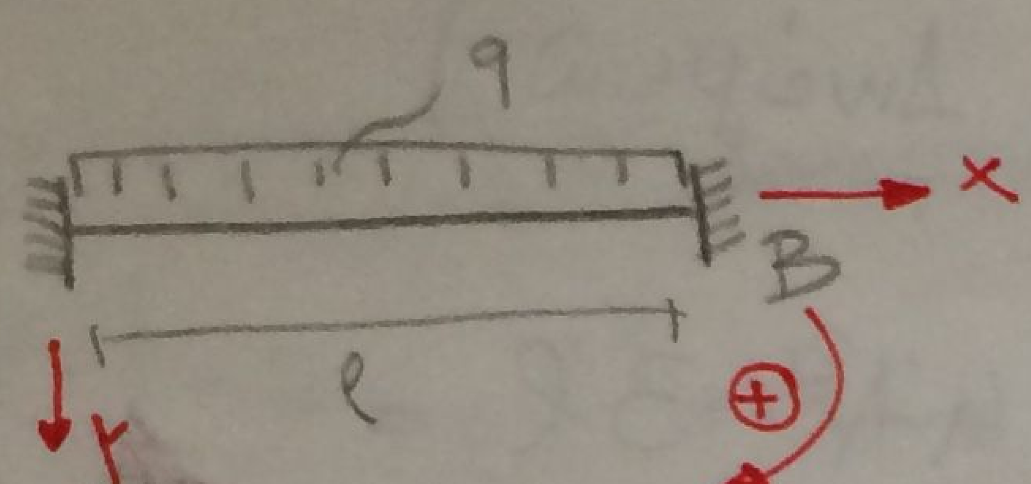
$\frac{\partial \Pi}{\partial A} = 0 \Leftrightarrow 6EI\ell^2 + 82EI\ell A + 28EIB - 9F\ell^2 = 0$

$\frac{\partial \Pi}{\partial B} = 0 \Leftrightarrow \frac{10EI}{\ell}B + 28EIA - 3F\ell = 0$

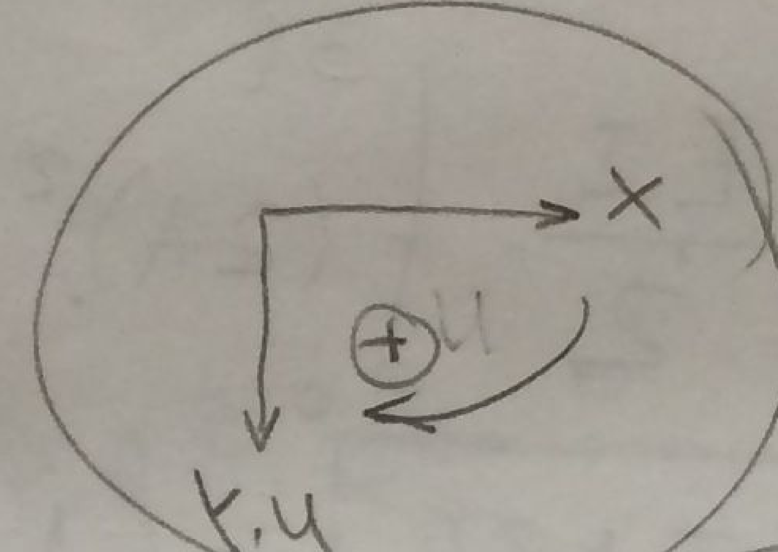
$A = \dots$
$B = \dots$

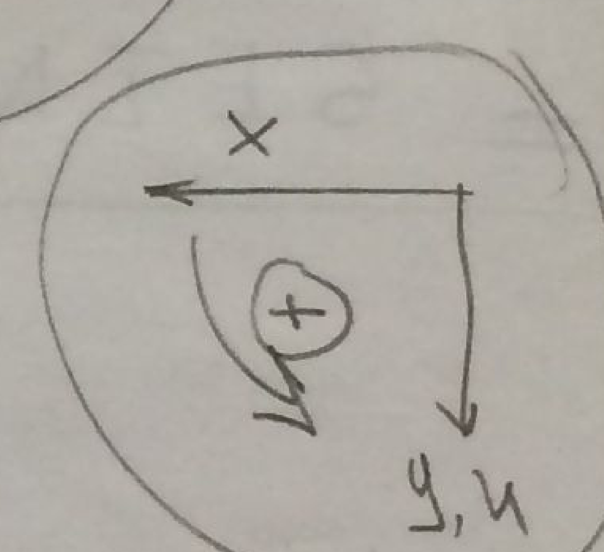
$u_{\text{σημ}\,\Gamma}(x=3\ell) = \dots$

3



$u(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$





- A πάκτωση { ορ. μετατόπιση = 0, κατ. μετατόπιση = 0, στροφή = 0
- B πάκτωση >> >>.

$u(x=0) = 0 \Leftrightarrow a_0 = 0$

$u'(x=0) = 0 \Leftrightarrow a_1 = 0$

$\boxed{u(x=\ell)=0}$ $\boxed{u'(x=\ell)=0}$

$2a_2\ell + 3a_3\ell^2 + 4a_4\ell^3 = 0$

$a_2\ell^2 + a_3\ell^3 + a_4\ell^4 = 0$

$\Leftrightarrow a_2 + a_3\ell + a_4\ell^2 = 0$

$2a_2 + 3a_3\ell + 4a_4\ell^2 = 0$

$a_2 = -a_3\ell - a_4\ell^2$

$-2a_3\ell - 2a_4\ell^2 + 3a_3\ell + 4a_4\ell^2 = 0$

$\Leftrightarrow a_3\cancel{\ell} = -2a_4\ell^{\cancel{2}} \Leftrightarrow$

$a_3 = -2a_4\ell$ (*)

$a_2 = 2a_4\ell^2 - a_4\ell^2$ (*)

$\Leftrightarrow a_2 = a_4\ell^2$

Αρχ.

$$u_{(x)} = a_4 \ell^2 x^2 - 2 a_4 \ell x^3 + a_4 \cdot x^4$$

$$U = \frac{EI}{2} \int_0^{\ell} \left(u''_{(x)} \right)^2 dx = \dots$$

$$(\text{Εργο Εξ. Δυναμεων}) = \int_0^{\ell} q_{(x)} \cdot u_{(x)} \cdot dx$$

$$= \int_0^{\ell} q \cdot u_{(x)} \, dx = \dots$$

$$\Pi = U - (\text{Εργο Εξ. Δυναμεων}) = \dots$$

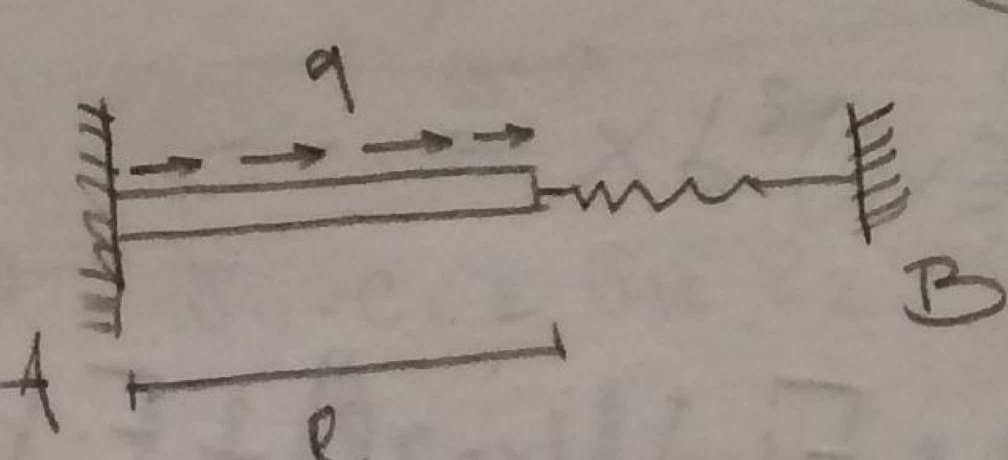
$$\frac{\partial \Pi}{\partial a_4} = 0 \Leftrightarrow a_4 = \dots$$

Αξονικό Φορτίο

$$u''_{(x)} = 2a_2$$

$$u'_{(x)} = a_1 + 2a_2 \cdot x$$

4



$$u_{(x)} = a_0 + a_1 x + a_2 \cdot x^2.$$

Απαιτήσεις $\left\{ \begin{array}{l} u_{(x=0)} = 0 \Leftrightarrow a_0 = 0 \end{array} \right\}$

$$\boxed{u_{(x)} = a_1 x + a_2 \cdot x^2} \quad \left\{ \begin{array}{l} u_{(x=\ell)} = 0 \\ u'_{(x=\ell)} = 0 \end{array} \right\}$$

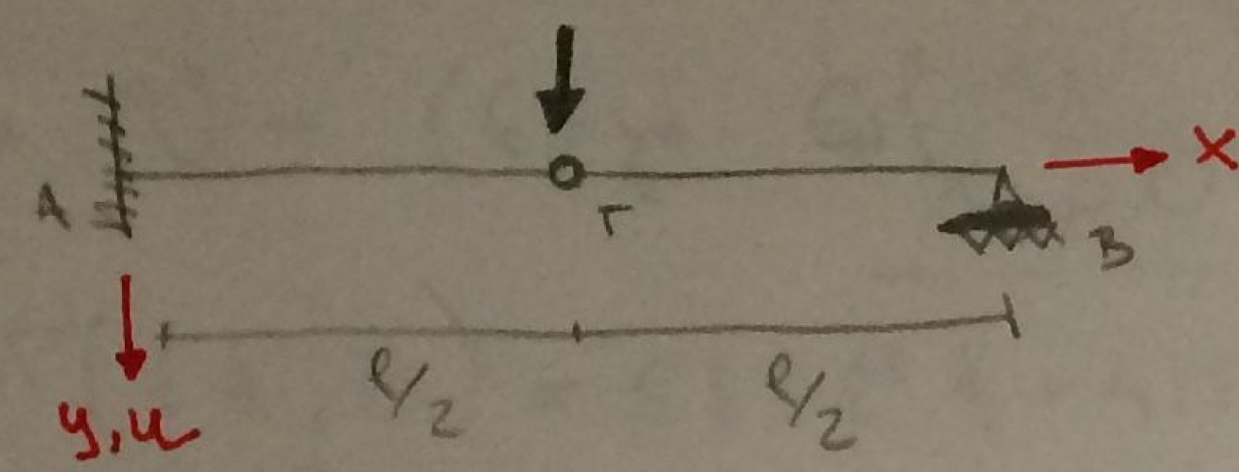
$$U = \int_0^{\ell} \frac{EA}{2} \epsilon_{xx}^2 \, dx \quad \text{οπου} \quad \epsilon_{xx} = \frac{\partial u}{\partial x} = a_1 + 2a_2 x$$

$$+ \frac{1}{2} K \cdot \left(u_{(x=\ell)} \right)^2 = U(a_1, a_2)$$

$$(\text{Εργο Εξ. Δυναμεων}) = \int_0^{\ell} q_{(x)} \cdot u_{(x)} \, dx = \dots$$

(8)

5



$u_1(x) = c_0 + c_1 \cdot x + c_2 \cdot x^2 + c_3 x^3 \,, \; 0 \le x \le \frac{\ell}{2}$ (7)

$u_2(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 \quad \frac{L}{2} \le x \le$

Α: Πάκτωση Β: αρθρωση.

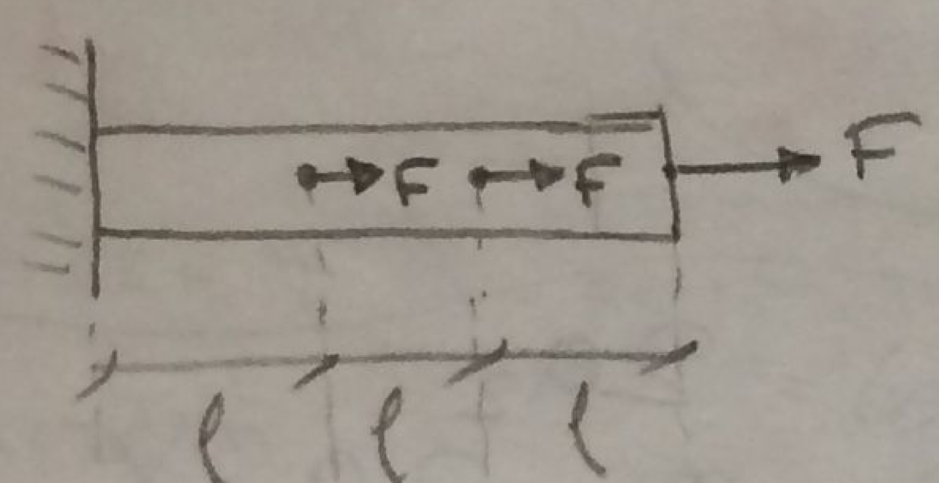
$u_1(x=0) = 0 \Leftrightarrow c_0 = 0$, $u'(x=0) = 0 \Leftrightarrow c_1 = 0$ $\quad u_1(x) = c_2 x^2 + c_3 x^3$

$u_2(x=\ell) = 0 \Leftrightarrow b_0 + b_1 \cdot \ell + b_2 \ell^2 + b_3 \ell^3 = 0$ (2)

$$\boxed{u_1(x=\ell/2) = u_2(x=\ell/2)}$$ (3)

Έργο Εξωτερικών Δυνάμεων $= P \cdot u_1(x=\ell/2) = P \cdot u_2(x=\ell/2)$.

6



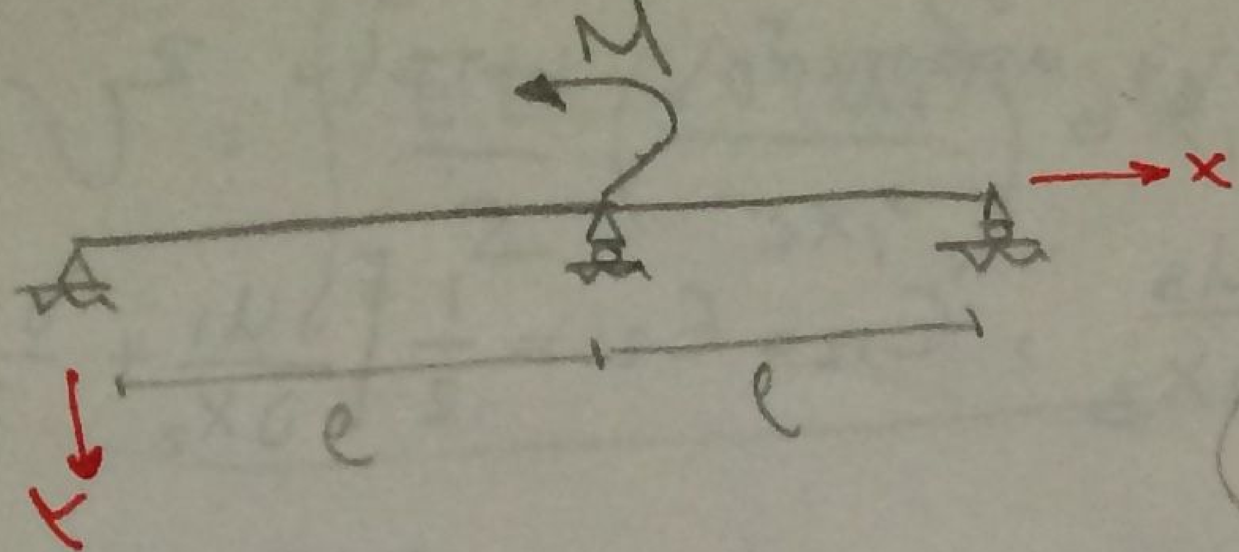
$u(x) = a_1 x + a_2 \cdot x^2$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = a_1 + 2 a_2 x$$

$$U = \frac{EA}{2} \int_0^{3\ell} (\varepsilon_{xx})^2 \, dx = ---$$

Έργο εξ. Δυνάμεων $= F \cdot u(x=\ell) + F \cdot u(x=2\ell) + F \cdot u(x=3$

7



$$y(x) = a\cdot\sin\left(\frac{\pi x}{\ell}\right)$$

Στροφή στο Β=;

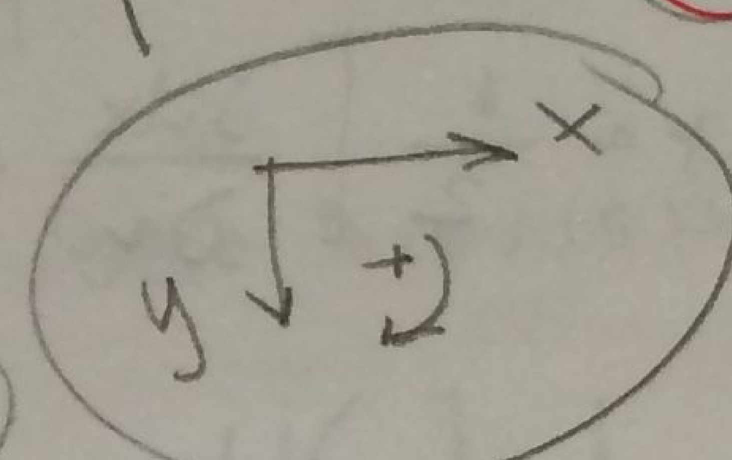
$y(x=0)=0$ ✓

$y(x=\ell)=0$ ✓

$y(x=2\ell)=0$ ✓

$$U = \frac{EI}{2}\int_0^{2\ell}\left(y''(x)\right)^2 dx$$

Έργο Εξωτερικών Δυνάμεων $= -M\cdot\varphi_B = -M\cdot y'(x=\ell)$



SOS

$$\Pi = \frac{EI}{2}\int_0^{2\ell}\left(y''(x)\right)^2 dx - \left[-M\cdot y'(x=\ell)\right]$$

$$\left.\begin{aligned}
T^{(1)} &= \sigma_{11}\cdot e_1 + \sigma_{12}\cdot e_2 + \sigma_{13}\cdot e_3\\
T^{(2)} &= \sigma_{12}\, e_1 + \sigma_{22}\, e_2 + \sigma_{23}\, e_3\\
T^{(3)} &= \sigma_{31}\cdot e_1 + \sigma_{32}\cdot e_2 + \sigma_{33}\cdot e_3
\end{aligned}\right\}$$

$$\begin{vmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma \end{vmatrix} = 0 \cdots \longrightarrow \boxed{\sigma^3 - I\,\sigma^2 + II\,\sigma - III = 0}$$

$\boxed{\text{Εξισώσεις Ισορροπίας}}$ SOS

$$\left(\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{21}}{\partial x_2} + \frac{\partial\sigma_{31}}{\partial x_3} = 0\right), \left(\frac{\partial\sigma_{12}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \frac{\partial\sigma_{32}}{\partial x_3} = 0\right), \left(\frac{\partial\sigma_{13}}{\partial x_1} + \frac{\partial\sigma_{23}}{\partial x_2} + \frac{\partial\sigma_{33}}{\partial x_3} = 0\right)$$

→ $\boxed{\text{Παραμόρφωση}}$: Εκφράζει την αλλαγή απόστασης μεταξύ 2 γειτονικών σημείων.

$$\varepsilon_{11}=\frac{\partial u_1}{\partial x_1},\quad \varepsilon_{22}=\frac{\partial u_2}{\partial x_2},\quad \varepsilon_{33}=\frac{\partial u_3}{\partial x_3},\quad \varepsilon_{12}=\varepsilon_{21}=\frac{1}{2}\left[\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right]$$

$$\varepsilon_{31}=\varepsilon_{13}=\frac{1}{2}\left[\frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3}\right],\quad \varepsilon_{32}=\varepsilon_{23}=\frac{1}{2}\left[\frac{\partial u_3}{\partial x_2}+\frac{\partial u_2}{\partial x_3}\right]$$

$$\theta = 2\cdot\varepsilon_{ij} \Leftrightarrow \boxed{\gamma = 2\cdot\varepsilon_{ij}}$$

η $\varepsilon_{xx}=\frac{\partial u_x}{\partial x},\quad \varepsilon_{yy}=\frac{\partial u_y}{\partial y},\quad \varepsilon_{zz}=\frac{\partial u_z}{\partial z}$.

$$\varepsilon_{xy}=\frac{1}{2}\left[\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right],\quad \varepsilon_{zy}=\frac{1}{2}\left[\frac{\partial u_z}{\partial y}+\frac{\partial u_y}{\partial z}\right]$$

$$\varepsilon_{zx}=\frac{1}{2}\left[\frac{\partial u_z}{\partial x}+\frac{\partial u_x}{\partial z}\right].$$

50%

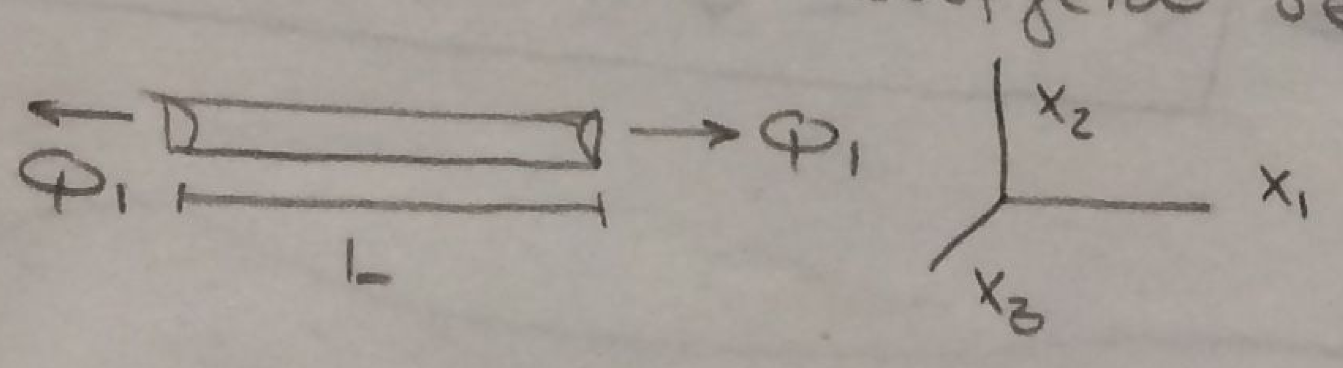
→ Εξισώσεις Συμβιβαστού

Όταν $u_3=0$ (επίπεδη περίπτωση) $\frac{\partial^2\varepsilon_{11}}{\partial x_2^2}=\frac{\partial^3 u_1}{\partial^2 x_2\,\partial x_1}$

$$\frac{\partial^2\varepsilon_{22}}{\partial x_1^2}=\frac{\partial^3 u_2}{\partial x_1^2\,\partial x_2},\quad 2\frac{\partial^2\varepsilon_{12}}{\partial x_1\partial x_2}=\frac{\partial^3 u_1}{\partial x_1\partial x_2^2}+\frac{\partial^3 u_2}{\partial x_2\partial x_1^2}$$

$$\frac{\partial^2\varepsilon_{11}}{\partial x_2^2}+\frac{\partial^2\varepsilon_{22}}{\partial x_1^2}=2\frac{\partial^2\varepsilon_{12}}{\partial x_1\partial x_2}$$

⊛ Παραμορφωσιακή ενέργεια σε ράβδο υπό αξονική φόρτιση



$\sigma_{11}=\frac{\Phi_1}{A},\quad \sigma_{ij}=0.$

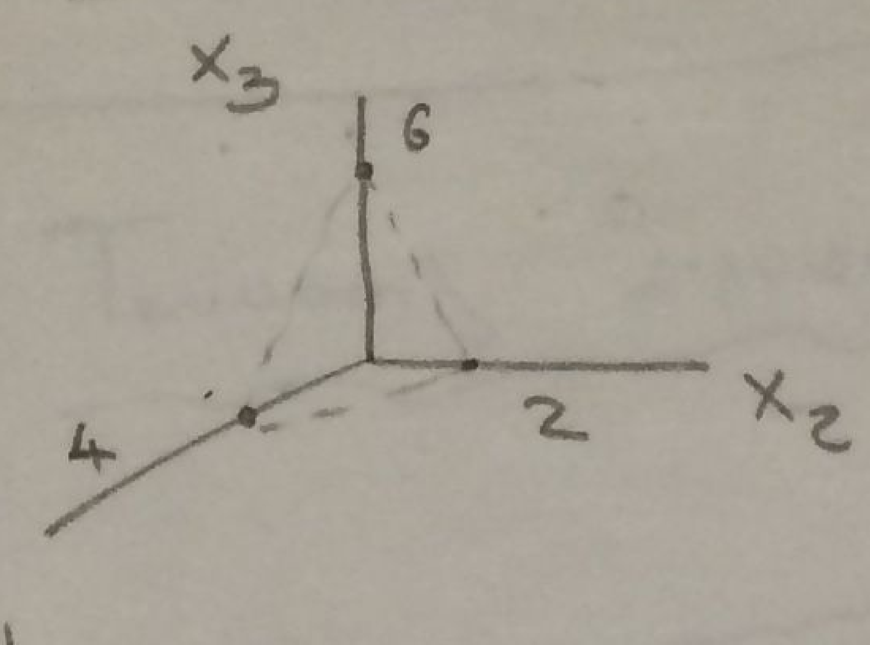
$$U=\int_V \frac{1}{2E}\cdot\sigma_{11}^2\,dV=\frac{1}{2E}\cdot\frac{\Phi_1^2}{A^2}\cdot\int_V dV=\frac{1}{2EA^2}\cdot\Phi_1^2\cdot A\cdot L$$

$$\Leftrightarrow U=\frac{\Phi_1^2\cdot L}{2EA}$$

δ) $U = \int_0^L \frac{EI}{2}\left(\frac{\partial^2 u_2}{\partial x_1^2}\right)^2 dx_1$

Θέση P: $\Sigma = \begin{pmatrix} 7 & -5 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

$T = ;$ διερχόμενο από το P // ABC $(3x_1+6x_2+2x_3=12$



$\boxed{T_i^{(n)} = \sigma_{ji}\cdot n_j}$

n_j: μοναδιαίο κάθετο διάνυσμα:

$\text{grad}\varphi = \nabla(3x_1+6x_2+2x_3-12)$

$= 3e_1+6e_2+2e_3$

$n = \frac{\text{grad}\varphi}{\|\text{grad}\varphi\|}$, $\quad n = \frac{3}{7}e_1 + \frac{6}{7}e_2 + \frac{2}{7}e_3$

$$\begin{bmatrix} 7 & -5 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}_{\underline{3\times3}} \begin{bmatrix} 3/7 \\ 6/7 \\ 2/7 \end{bmatrix}_{\underline{3\times1}} = \begin{bmatrix} 3 - \frac{30}{7} + 0 \\ -\frac{15}{7} + \frac{18}{7} + \frac{2}{7} \\ \frac{6}{7} + 4/7 \end{bmatrix} = \begin{bmatrix} -9/7 \\ 5/7 \\ 10/7 \end{bmatrix}$$

$$\boxed{T = -\frac{9}{7}e_1 + \frac{5}{7}e_2 + \frac{10}{7}e_3}$$

$\Sigma = \begin{pmatrix} 3x_1x_2 & 5x_2^2 & 0 \\ 5x_2^2 & 0 & 2x_3 \\ 0 & 2x_3 & 0 \end{pmatrix}$ $\quad T = ;\quad \dfrac{x_2^2+x_3^2-4=0}{\text{στο } P(2,1,\sqrt{3})}$

$\text{grad}f = 2x_2\cdot e_2 + 2x_3\cdot e_3 \quad \|\text{grad}f\| = \sqrt{4+12} = 4$

$\text{grad}f_P = 2\cdot e_2 + 2\sqrt{3}\,e_3$ Άρα $n = \frac{2}{4}e_2 + \frac{2\sqrt{3}}{4}e_3$

$$\boxed{\eta = \frac{1}{2} e_2 + \frac{\sqrt{3}}{2} \cdot e_3}$$

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Στο P $\quad \Sigma_p = \begin{pmatrix} 6 & 5 & 0 \\ 5 & 0 & 2\sqrt{3} \\ 0 & 2\sqrt{3} & 0 \end{pmatrix}$

$$\Sigma_p \cdot \eta = \begin{pmatrix} 6 & 5 & 0 \\ 5 & 0 & 2\sqrt{3} \\ 0 & 2\sqrt{3} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \frac{1}{2} \\ \sqrt{3}/2 \end{pmatrix} = \dots$$

(*)

$$\sigma_{ij} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

• Τιμές Κυρίων Τάσεων

$$\begin{vmatrix} 3-\sigma & 1 & 1 \\ 1 & -\sigma & 2 \\ 1 & 2 & -\sigma \end{vmatrix} = 0 \Leftrightarrow (\sigma+2)\cdot(\sigma-4)\cdot(\sigma-1) = 0.$$

$$\sigma^{(I)} = -2, \ \sigma^{(II)} = 4, \ \sigma^{(III)} = 3.$$

(*) (I), (II), (III)

$$\sigma_{ij} = \begin{pmatrix} 6 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

$I = \sigma_{11} + \sigma_{22} + \sigma_{33} = 20 = I$

$II = \frac{1}{2}\left[\sigma_{ii}\cdot\sigma_{jj} - \sigma_{ij}\cdot\sigma_{ji}\right]$

$(\sigma_{11}\cdot\sigma_{22}) + \cancel{(\sigma_{22}\cdot\sigma_{22})} + (\sigma_{33}\cdot\sigma_{22})$

$+(\sigma_{33}\cdot\sigma_{11}) = 36 + \cancel{36} + 48 + 48$

$(\sigma_{ij}\sigma_{ji}) = (\sigma_{12}\cdot\sigma_{21})\cdot(\sigma_{31}\cdot\sigma_{13})\cdot(\sigma_{23}\cdot\sigma_{32}) =$

$= 9 + 0 + 0 = 9$

$II = 36 + 48 + 48 - 9 = 123 = II$

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

$III = |\sigma_{ij}|$

$i=1 \leftarrow 2,3$

$i=2 \leftarrow 1,3$

$i=3 \leftarrow 2,1$

$\sigma_{12} = -3 \quad \sigma_{13} = 0$

$\sigma_{21} = -3 \quad \sigma_{23} = 0$

$\sigma_{32} = 0 \quad \sigma_{31} = 0$

(*) $u = (x_1 - x_3)^2 \cdot e_1 + (x_2 + x_3)^2 \cdot e_2 - x_1 \cdot x_2 \cdot e_3$

(Τανυστή Τροπών) + (Τανυστή Στροφών) στο $P(0,2,-1)$.

$$\begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \cdots \\ \cdots & \frac{\partial u_2}{\partial x_2} & \cdot \\ & & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

Τανυστής Στροφών: $W_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right]$

Κεφάλαιο 3

→ Επίπεδη Ένταση και Επίπεδη Παραμόρφωση:

α) Επίπεδη Ένταση

$\sigma_{zz} = \sigma_{zx} = \sigma_{zy} = 0$

$\epsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - v \cdot \sigma_{yy}\right]$

$\epsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - v \cdot \sigma_{xx}\right]$

$\epsilon_{zz} = \frac{1}{E}\left[-v(\sigma_{xx} + \sigma_{yy})\right]$

$\epsilon_{xy} = \frac{\sigma_{xy}}{2G}$, $\epsilon_{xz} = \epsilon_{zx} = 0$.

β) Επίπεδη Παραμόρφωση

$u_z = \frac{\partial u_x}{\partial z} = \frac{\partial u_y}{\partial z} = 0$

$\epsilon_{zz} = \epsilon_{xz} = \epsilon_{yz} = 0$

$\frac{1}{E}\left[\sigma_{zz} - v(\sigma_{yy} + \sigma_{xx})\right] = 0$

$\Leftrightarrow \boxed{\sigma_{zz} = v(\sigma_{yy} + \sigma_{xx})}$

→ Τασική Συνάρτηση Airy:

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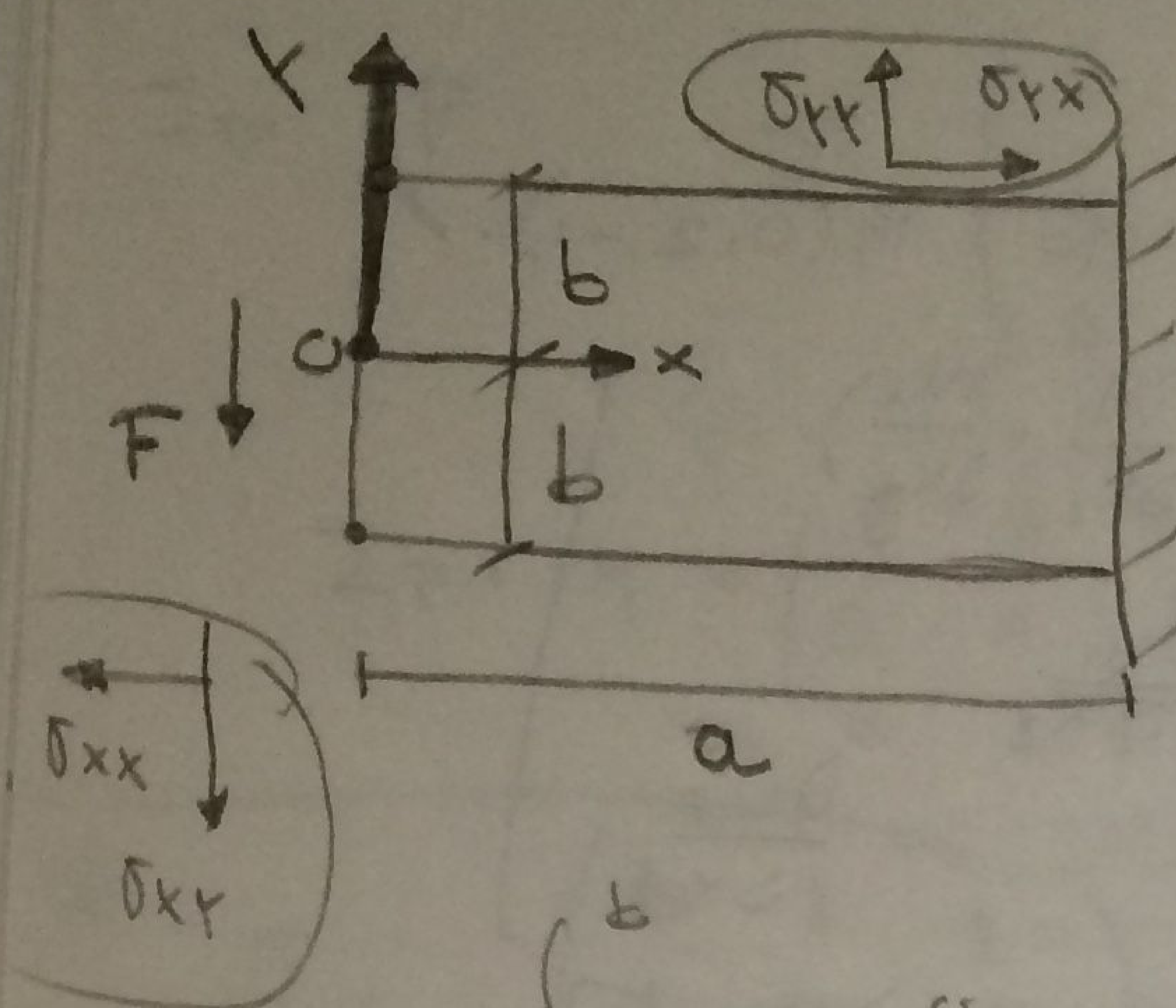
$\nabla^4 \Phi = 0$ {Διαρμονική Εξίσωση}.

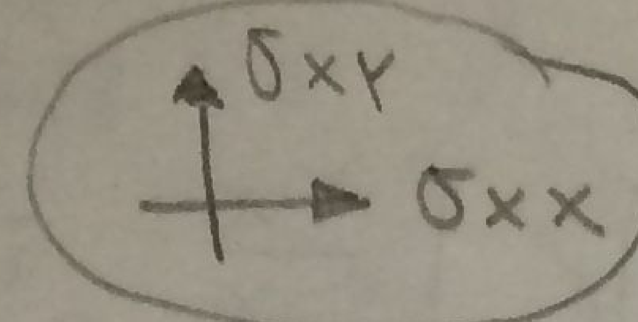
$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\left(\frac{\partial^4}{\partial x^2 \partial y^2}\right) + \left(\frac{\partial^4}{\partial y^4}\right).$$

$$\sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}.$$

- **Προβλημα Καμψης Δοκού**



- Συνοριακές Συνθηκες:



$$\sigma_{xy}=0,\ y=\pm b$$

$$\sigma_{yy}=0,\ y=\pm b$$

$$\sigma_{xx}=0,\ x=0$$

$$\int_{-b}^{b}\sigma_{xy}\,dy=F \quad \text{για } x=0$$

$$\int_{-b}^{b}\sigma_{xx}\,dy=0,\ x=a$$

- **Πολικές Συντεταγμένες**

* Αξονοσυμμετρικά Προβλήματα

Γεωμετρία + Φορτιση ανεξαρτητες της γωνίας θ.

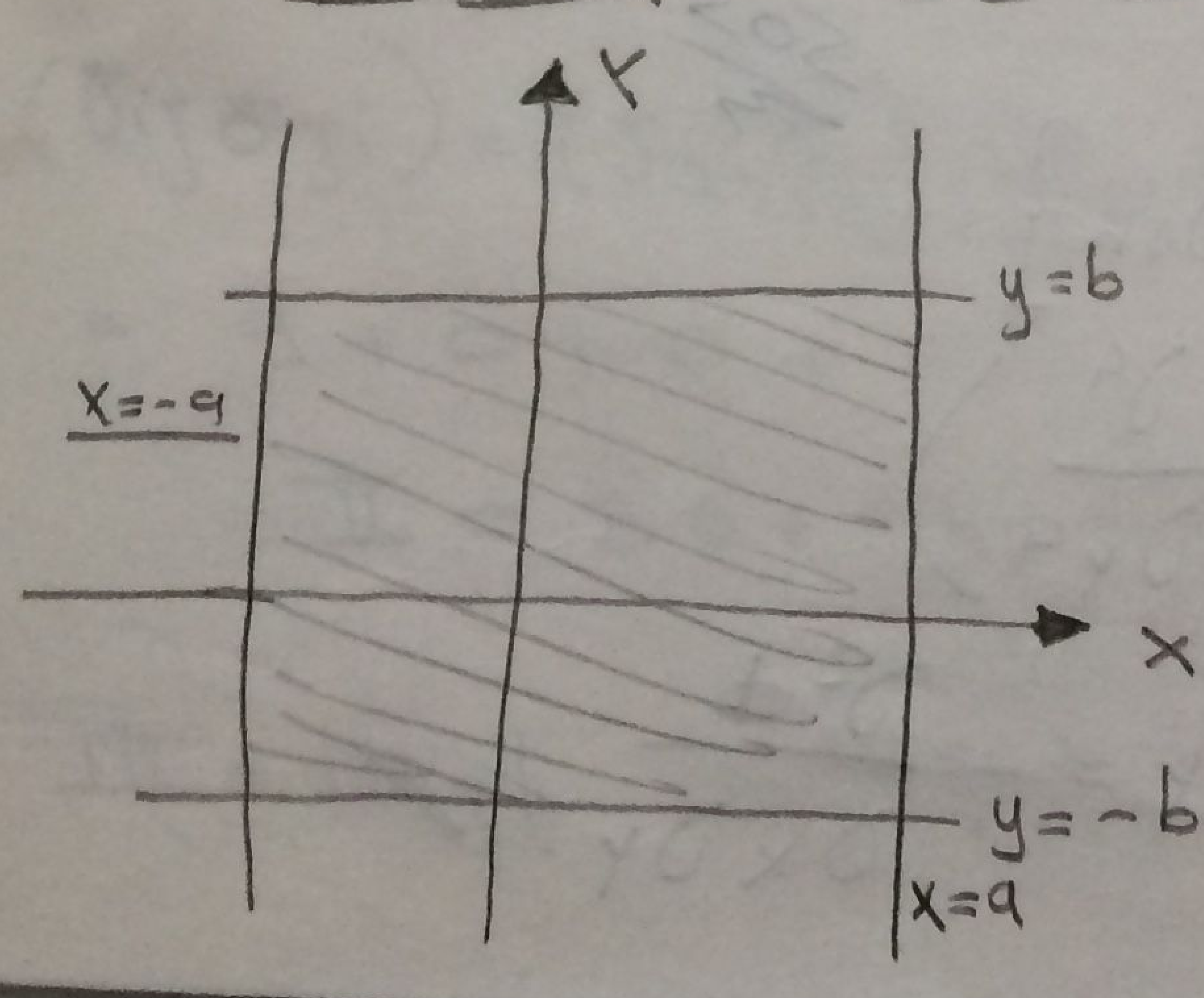
$$\Phi \equiv \Phi(r).$$

Ασκησεις

SOS/Μ

1) $u_x=-a_1(y^2+\nu x^2)$, $u_y(x,y)=2a_1 x y$.

$\boxed{\sigma_{zz}=0}$ Ζητουνται οι τάσεις = ;



$$\sigma_{xx}=\frac{E}{1-\nu^2}\left[\epsilon_{xx}+\nu\cdot\epsilon_{yy}\right]$$

$$\sigma_{yy}=\frac{E}{1-\nu^2}\left(\epsilon_{yy}+\nu\cdot\epsilon_{xx}\right)$$

$$\sigma_{xy}=\frac{E}{1+\nu}\epsilon_{xy}.$$

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = -2\nu a_1 x \quad , \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 2a_1 x .$$

$$\varepsilon_{xy} = \frac{1}{2}\left[\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right] = \frac{1}{2}\left[-2a_1 y + 2a_1 y\right]$$

$$\Leftrightarrow \varepsilon_{xy} = 0$$

Οπότε $\sigma_{xx} = \frac{E}{1-\nu^2}\left[-2a_1 \nu x + \nu \cdot 2a_1 x\right] = 0$

$$\boxed{\sigma_{yy} = \frac{E}{1-\nu^2}\left[2a_1 x - 2a_1 x \nu^2\right]}$$

$$\sigma_{xy} = \frac{E}{1+\nu} \cdot 0 = 0$$

---

2) $\Phi(x) = \begin{cases} a_1 x^2 & , 0 < x < 1 \\ a_1(2x-1) & , 1 < x < 2 \end{cases}$ αποτελεί λύση επίπεδης ελαστικής κατάστασης $x=0, x=2, y=0, y=2$

$\frac{50}{//}$

$\boxed{\text{Λύση Απόδειξη;}}$

i) Μονάδες $\Phi(x)$ είναι διαφορετικές στα δύο διαστήματα.

ii) $\sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2}$, $\sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2}$, $\sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}$.

$$\frac{\partial \Phi}{\partial x} = \begin{Bmatrix} 2a_1 x \\ 2a_1 \end{Bmatrix} \quad , \quad \frac{\partial^2 \Phi}{\partial x^2} = \begin{Bmatrix} 2a_1 \\ 0 \end{Bmatrix}$$

$$\frac{\partial \Phi}{\partial y} = 0 \quad , \quad \frac{\partial^2 \Phi}{\partial y^2} = 0 \qquad \sigma_{xx} = \{0\} \quad \sigma_{yy} = \begin{Bmatrix} 2a_1 \\ 0 \end{Bmatrix}$$

$$\sigma_{xy} = \{0\}.$$

$$\varepsilon_{xx} = \frac{1}{E}\left[-\nu \cdot \sigma_{yy}\right] = \begin{Bmatrix} -\frac{2a_1 \nu}{E} \\ 0 \end{Bmatrix}$$

$$\varepsilon_{yy} = \frac{1}{E}\begin{bmatrix} 2a_1 \\ 0 \end{bmatrix} = \begin{Bmatrix} \frac{2a_1}{E} \\ 0 \end{Bmatrix}$$

3) $\Phi(x) = a_1(x^4 - 3x^2y^2)$

i) Ικανοποιεί την $\nabla^4\phi = 0$

ii) Τάσεις; (ox, oy, x=1, y=1).

iii) Εξέταση Ισορροπίας ως προς το (x=y=0).

i) $$\nabla^4\phi = \left(\frac{\partial^4\phi}{\partial x^4}\right) + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4}$$

$$\frac{\partial\phi}{\partial x} = a_1(4x^3 - 6xy^2), \quad \frac{\partial^2\phi}{\partial x^2} = a_1(12x^2 - 6y^2)$$

$$\frac{\partial^3\phi}{\partial x^3} = a_1(24x), \quad \frac{\partial^4\phi}{\partial x^4} = 24a_1$$

$$\frac{\partial^3\phi}{\partial x^2\partial y} = a_1(-12y) \quad \frac{\partial^4\phi}{\partial x^2\partial y^2} \quad -12a_1 = \frac{\partial^4\phi}{\partial x^2\partial y^2}$$

$$\frac{\partial\phi}{\partial y} = a_1(-6x^2y), \quad \frac{\partial^2\phi}{\partial y^2} = a_1(-6x^2), \quad \frac{\partial\phi^3}{\partial y^3} = 0, \quad \frac{\partial\phi^4}{\partial y^4} = 0$$

$$\nabla^4\phi = 24a_1 - 24a_1 + 0 = 0 \checkmark$$

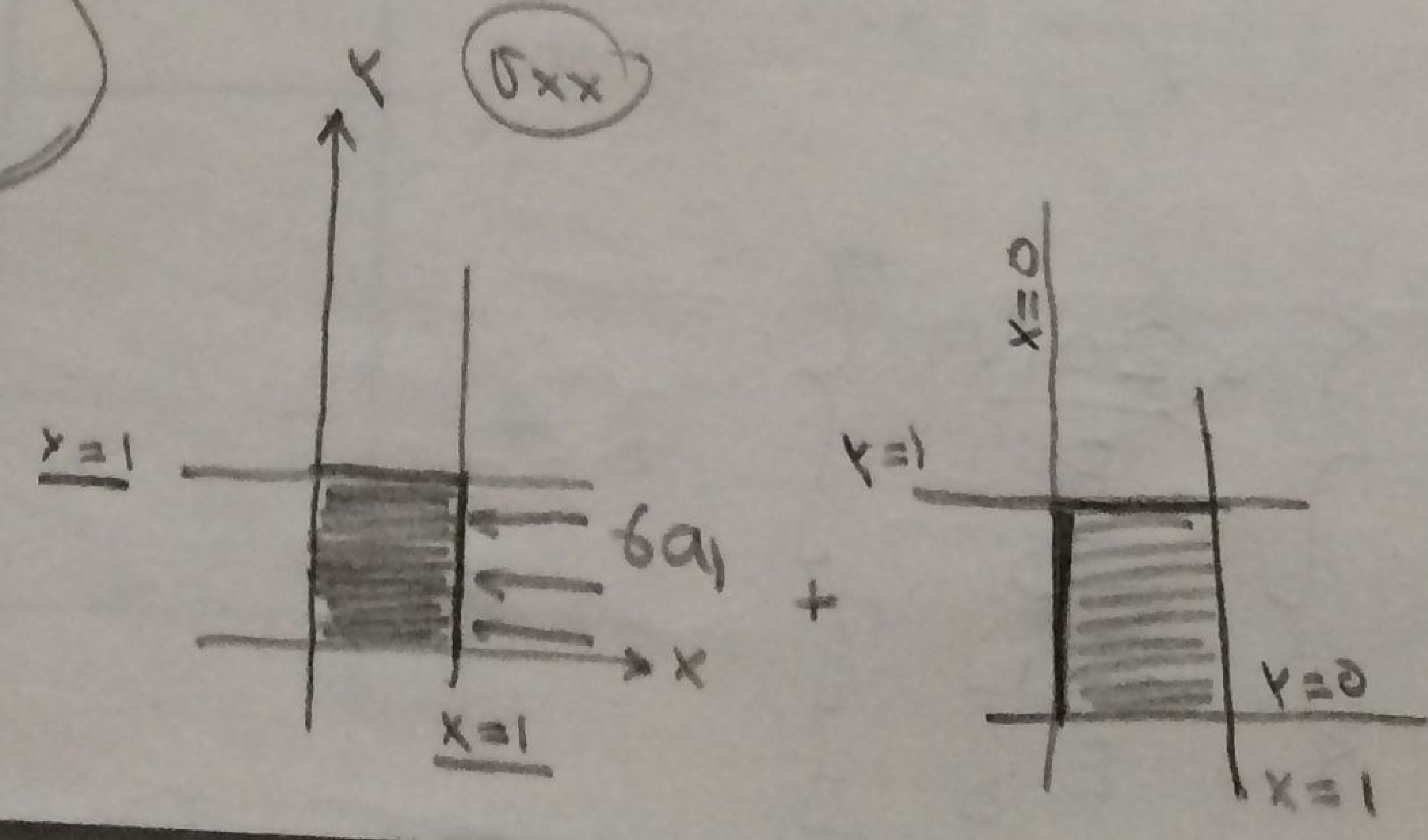
$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = -6a_1x^2, \quad \sigma_{yy} = a_1(12x^2 - 6y^2)$$

$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -\frac{\partial}{\partial y}\left(a_1(4x^3 - 6xy^2)\right)$$

$$= -a_1(-12xy) = 12a_1xy.$$

x=0 x=1

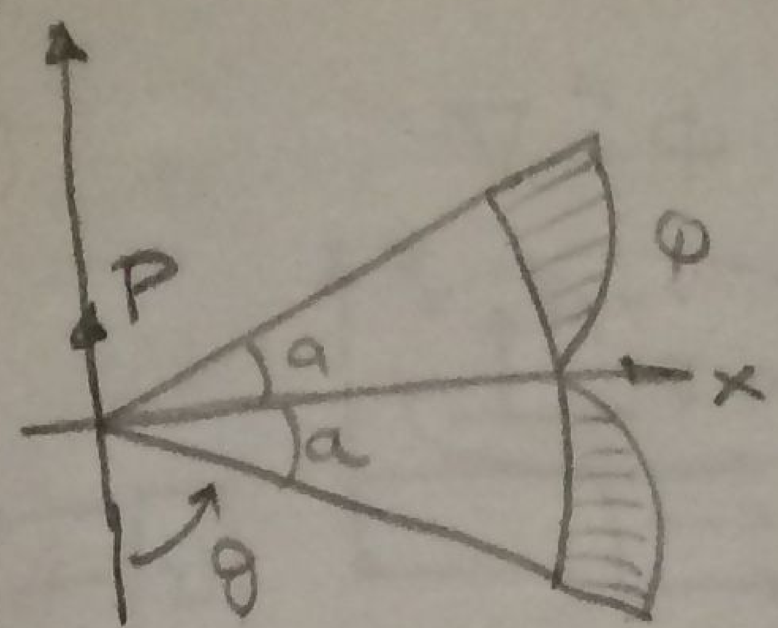
y=0 y=1



y=1, x=0 $6a_1$

x=1 y=0 $12a_1$

4) $\sigma_{rr} = 2a_1 \frac{\cos\theta}{r}$, $\sigma_{\theta\theta} = \sigma_{r\theta} = 0$



$$\theta = \frac{\pi}{2} + \beta \qquad \cos\theta = -\sin\beta$$

$$\sigma_{rr} = -2a_1 \frac{\sin\beta}{r}$$

$$0 = P + \int_{-a}^{a} (\sigma_{rr}) \cdot r \cdot d\beta \, \sin\beta = \dots$$

5) Ασθενείς Συνοριακές Συνθήκες στο $x=0$.

$$\Phi(x,y) = Ax^4y + Bx^2y^3 + Cy^5 + Dx^2y + Ey^3$$

$\sigma_{yy} = 0$, $\sigma_{xy} = 0$

$$\sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}$$

$$\nabla^4 \Phi = 0 \quad \acute{\eta} \quad \nabla^2 (\sigma_{xx} + \sigma_{yy}) = 0$$

Μεσοδοκία

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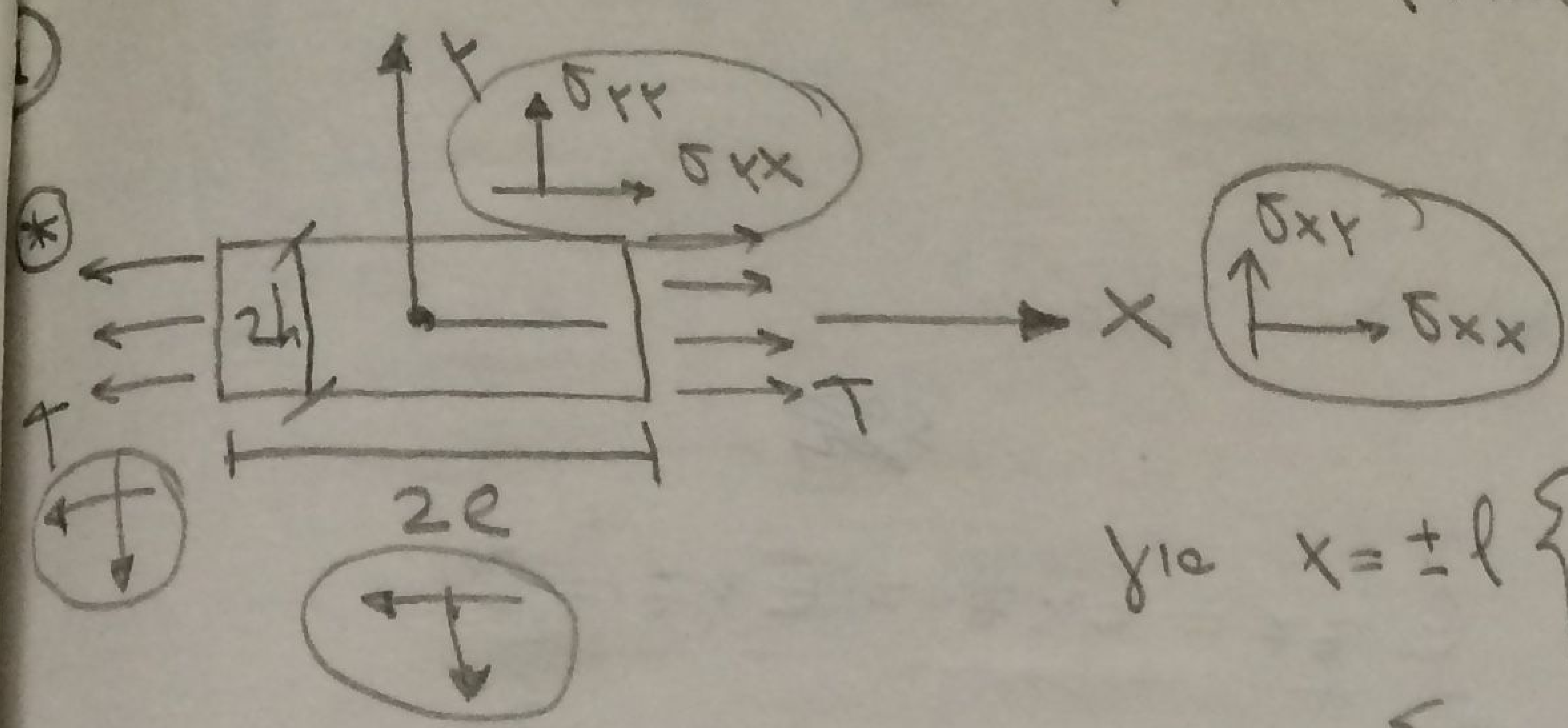

για $x = \pm l \begin{cases} \sigma_{xx} = T \\ \sigma_{xy} = 0 \end{cases}$

για $y = \pm h \begin{cases} \sigma_{yy} = 0 \\ \sigma_{yx} = 0 \end{cases}$

$\boxed{\Phi = a \cdot y^2}$ $\quad \sigma_{xx} = 2a = \frac{\partial^2 \Phi}{\partial y^2}$

$\sigma_{yy} = 0$ Οπότε $2a = T \Leftrightarrow a = \frac{T}{2}$

$\sigma_{xy} = 0$

$$\boxed{\Phi = \frac{T}{2} y^2}$$

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \epsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \epsilon_{xy} = \frac{1}{2}\left[\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right]$$

Για $a = T/2 \Rightarrow \begin{cases} \sigma_{xx} = T \\ \sigma_{yy} = \sigma_{xy} = 0 \end{cases}$

$$\sigma_{xx} = E \cdot \varepsilon_{xx} \Leftrightarrow T = E \cdot \varepsilon_{xx} \Leftrightarrow \epsilon_{xx} = \frac{T}{E} \Leftrightarrow$$

$$\frac{\partial u_x}{\partial x} = \frac{T}{E} \Leftrightarrow \partial u_x = \frac{T}{E} dx \Leftrightarrow \boxed{u_x = \frac{T}{E} \cdot x + f(y)}$$

$$\epsilon_{yy} = \frac{1}{E}[-v \cdot T] \Leftrightarrow \boxed{u_y = -\frac{vT}{E} \cdot y + g(x)}$$

$$\epsilon_{xy} = \frac{1}{2}\left[\frac{df}{dy} + \frac{dg}{dx}\right] = 0 \Leftrightarrow \frac{df}{dy} + \frac{dg}{dx} = 0 \Leftrightarrow \frac{df}{dy} = -\frac{dg}{dx} = C$$

$$f(y) = C \cdot y + C_1, \quad g(x) = -C \cdot x + C_2.$$

⊛ $\Phi = x^4 y + 4x^2 y^3 - y^5$.

Δίζ → κατάσταση επίπεδης παραμόρφωσης με $\nu = 0{,}25$.

Για να δω αν είναι κατάλληλη πρέπει

$$\boxed{\nabla^4 \Phi = 0}$$

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Επίπεδη Παραμόρφωση $\quad \epsilon_{zz} = \epsilon_{xz} = \epsilon_{yz} = 0$

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix} \qquad \sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2}$$

$$\sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}$$

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1) $\epsilon_{xx} = a_1 x^3$
$\epsilon_{yy} = a_1 x^2 y$
$\epsilon_{xy} = \dfrac{a_1 x y^2}{2}$

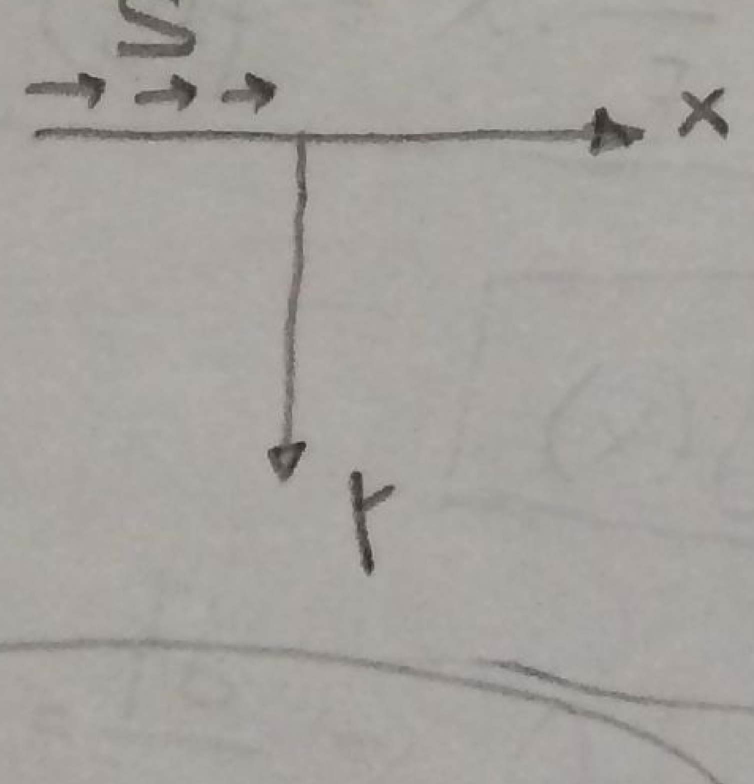
2) $\sigma_{xx} = a_1 x + a_2 y$
$\sigma_{yy} = a_3 x + a_4 y$
$\sigma_{xy} = -a_4 x - a_1 y$

3) $\sigma_{xx} = -a_1 x^2 y$
$\sigma_{yy} = -a_1 y^3/3$
$\sigma_{xy} = -a_1 x y^2$

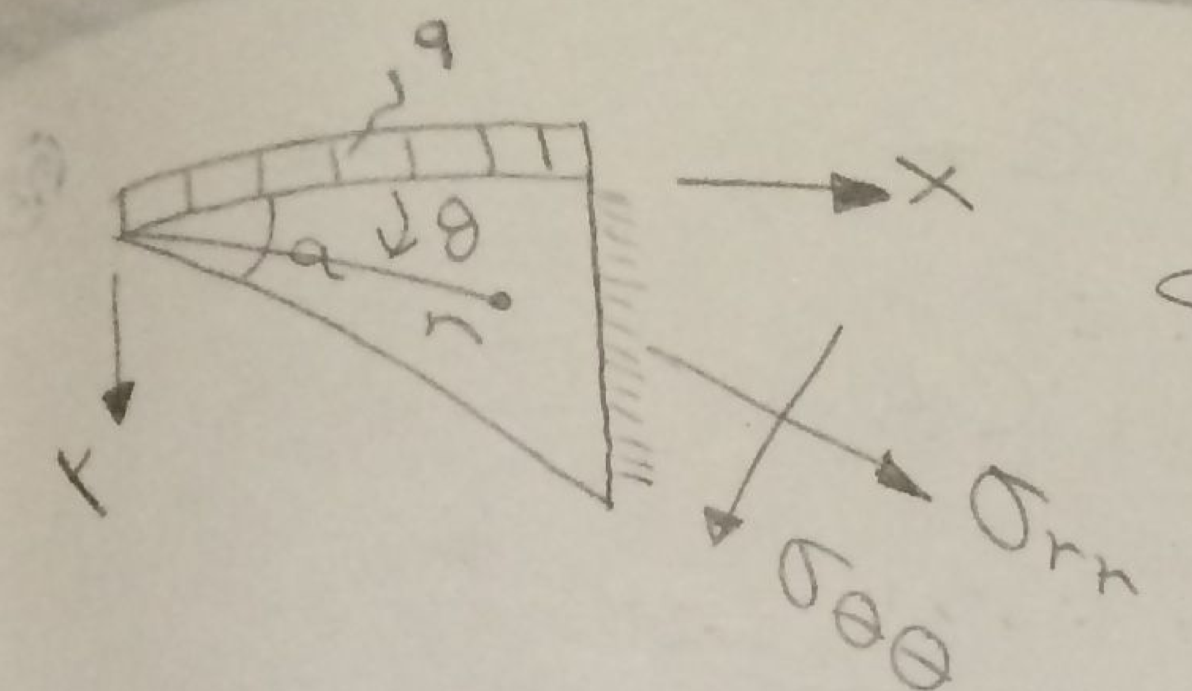
Κατάλληλες λύσεις επιπέδων προβλημάτων

Πρέπει να ικανοποιούν Ισορροπία + Συμβιβαστό

⊛ $\Phi = \dfrac{S}{\pi}\left[\dfrac{1}{2} y \ln(x^2+y^2) + xy \cdot \arctan\left(\dfrac{y}{x}\right) - y^2\right]$



$\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$

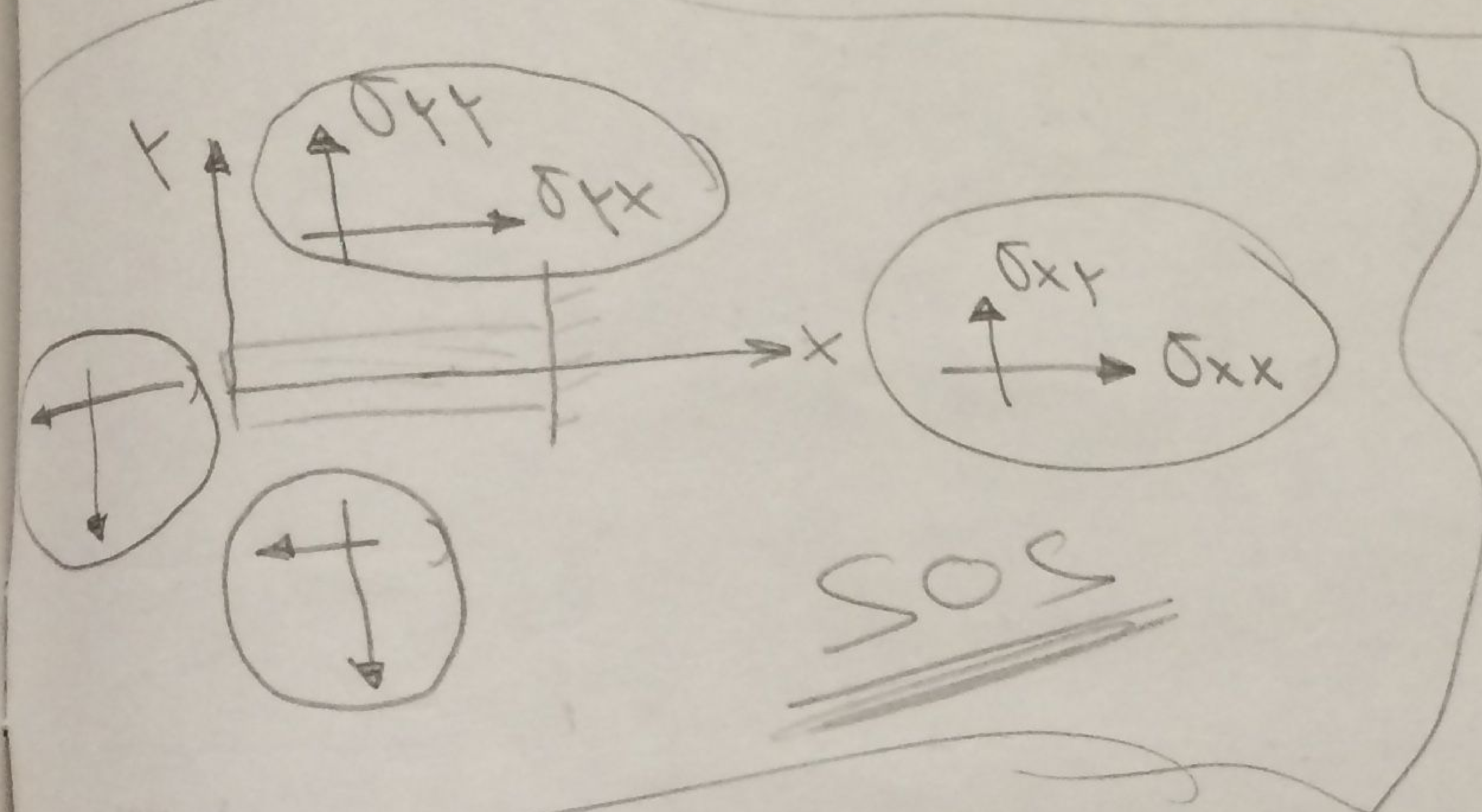


$$\Phi = c\left[r^2(a-\theta) + r^2 \sin\theta \cdot \cos\theta - r^2 \cos^2\theta \cdot \tan a\right]$$

$\underline{\theta = 0}$ $\begin{Bmatrix} \sigma_{rr} = 0 \\ \tau_{r\theta} = q \end{Bmatrix}$

$\underline{\theta = a}$ $\begin{Bmatrix} \sigma_{rr} = 0 \\ \sigma_{\theta\theta} = 0 \end{Bmatrix}$

Διότι δεν υπάρχει φορτίο



$\begin{cases} \sigma_{xx} = a_1 \cdot y \\ \sigma_{yy} = a_2 \cdot x \end{cases}$ $\quad 0 \le x \le a, \quad 0 \le y \le b$

i) Τασική Έκφραση για την Φ SOS

ii) $\sigma_{xy} = ;$

iii) Τασικές Εκφράσεις για μετατοπίσεις

$$\sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2} \Leftrightarrow a_1 \cdot y = \frac{\partial^2 \Phi}{\partial y^2} \Leftrightarrow \frac{\partial \Phi}{\partial y} = \frac{a_1 \cdot y^2}{2} + F_1(x)$$

$$\boxed{\Phi(x,y) = \frac{a_1 \cdot y^3}{6} + F_1(x) \cdot y + F_2(x)} \; (*)$$

$$\frac{\partial \Phi}{\partial x} = F_1'(x) \cdot y + F_2'(x), \quad \frac{\partial^2 \Phi}{\partial x^2} = \boxed{F_1''(x) \cdot y + F_2''(x) = a_2 \cdot x}$$

$\underline{\text{Οπότε}}$ $F_1''(x) = 0 \Leftrightarrow F_1'(x) = C$

$F_1(x) = Cx + C_2.$