Θεμα 1°

α) $(1+t^2)y' + 4ty = 2(1+t^2)^{-2}$

$$y' + \frac{4t}{1+t^2}y = 2(1+t^2)^{-3}$$

Ομογενής $\quad y' + \frac{4t}{1+t^2}y = 0 \Rightarrow$

$$\frac{dy}{dt} = -\frac{4t}{1+t^2}y \Rightarrow \left(\frac{1}{y}\right)dy = -2\cdot\frac{2t}{1+t^2}dt \Rightarrow$$

$$\ln y = -2\ln(1+t^2) + c \Rightarrow \ln y = \ln(1+t^2)^{-2} + c$$

$$y_0 = e^{\ln(1+t^2)^{-2}+c} \Rightarrow y_0 = e^c\cdot(1+t^2)^{-2} \Rightarrow y_0 = c\cdot(1+t^2)^{-2}$$

Εστω μια μερική λυση της Δ.Ε. με τη μορφή $y_\mu = c\cdot e^{\int p(t)dt}$

$$y_\mu = c(t)\cdot e^{-\int\frac{t4}{1+t^2}dt} = c(t)\cdot e^{-2\ln(1+t^2)} = c(t)(1+t^2)^{-2}$$

$$y_\mu' = c'(t)\cdot(1+t^2)^{-2} + c(t)\cdot(-2)(1+t^2)^{-3}\cdot 2t$$

$$c'(t)\cdot(1+t^2)^{-2} + (-4t)\cdot c(t)\cdot(1+t^2)^{-3} + c(t)(1+t^2)^{-2}\cdot\frac{4t}{1+t^2} = 2(1+t^2)^{-3}$$

$$c'(t)\cdot(1+t^2)^{-2} \cancel{-4t\,c(t)(1+t^2)^{-3}} + \cancel{4t\,c(t)(1+t^2)^{-3}} = 2(1+t^2)^{-3}$$

$$c'(t) = \frac{2}{1+t^2} \Rightarrow c(t) = 2\arctan t$$

οποτε $\quad y_\mu = 2\arctan t\,(1+t^2)^{-2}$

$$y = y_0 + y_\mu = c(1+t^2)^{-2} + 2\arctan t\,(1+t^2)^{-2}$$

$$= (1+t^2)^{-2}(c + 2\arctan t)$$

β) $\dfrac{dy}{dx} = \dfrac{x^2 - 3y^2}{2xy} \Rightarrow y' \cdot 2xy = x^2 - 3y^2 \Rightarrow$

$$(3y^2 - x^2) + (2xy) \cdot y' = 0$$

Έλεγχος εαν είναι ακριβής:

$$\left.\begin{array}{l} M = (3y^2 - x^2) \longrightarrow M_y = 6y \\ N = (2xy) \longrightarrow N_x = 2y \end{array}\right. \qquad M_y \neq N_x \rightarrow \text{όχι ακριβής}$$

Έστω $\mu = f(x)$ τέτοια ώστε $\dfrac{\partial \mu M}{\partial y} = \dfrac{\partial \mu N}{\partial x}$

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu \Rightarrow \frac{d\mu}{dx} = \frac{6y - 2y}{2xy}\mu \Rightarrow$$

$$\left(\frac{1}{\mu}\right) d\mu = \frac{4y}{2xy}\, dx \Rightarrow \left(\frac{1}{\mu}\right) d\mu = \left(\frac{2}{x}\right) dx \Rightarrow \ln\mu = 2\ln x$$

$$\Rightarrow \mu(x) = x^2$$

οπότε $\mu \cdot M(x,y) + \mu \cdot N(x,y)\, y' = 0 \Rightarrow (3y^2x^2 - x^4) + (2x^3y)\, y' = 0$

$$\left.\begin{array}{l} M = 3y^2x^2 - x^4 \longrightarrow M_y = 6yx^2 \\ N = 2x^3y \longrightarrow N_x = 6x^2y \end{array}\right\} \quad M_y = N_x \rightarrow \text{ακριβής}$$

οπότε υπάρχει $F(x,y)$ τέτοια ώστε $\dfrac{\partial F}{\partial x} = M(x,y)$ και $\dfrac{\partial F}{\partial y} = N(x,y)$

$$\frac{\partial F}{\partial y} = 2x^3y \Rightarrow F = x^3y^2 + c(x)$$

$$\frac{\partial F}{\partial x} = M \Rightarrow \cancel{3x^2y^2} + c'(x) = \cancel{3x^2y^2} - x^4 \Rightarrow c'(x) = -x^4$$

$$c(x) = -\frac{1}{5}x^5$$

$F(x,y) = C \Rightarrow x^3y^2 - \dfrac{1}{5}x^5 = C$ (Λύση σε πεπλεγμένη μορφή)

$$y^2 = \frac{1}{5}x^2 + \frac{C}{x^3}$$

$$y = \pm\sqrt{\frac{1}{5}x^2 + \frac{C}{x^3}}$$

ΘΕΜΑ 2

β) $y'' + 6y' + 9y = e^{-3t} + \cos 3t$

ομογενής: $y'' + 6y' + 9y = 0$

$r^2 + 6r + 9 = 0 \Rightarrow (r+3)^2 = 0 \Rightarrow r = -3$ διπλή ρίζα

$y_1 = e^{-3t}$

$y = v(t)\cdot y_1(t) = v(t)\cdot e^{-3t}$

$y' = v'(t)\cdot e^{-3t} + (-3)\, v(t)\, e^{-3t}$

$y'' = v''(t)\, e^{-3t} + (-3) v'(t)\cdot e^{-3t} + (-3) v'(t)\cdot e^{-3t} + 9 v(t)\cdot e^{-3t}$

$v''(t)\cdot e^{3t} \cancel{- 6v'(t)\cdot e^{-3t}} + \cancel{9v(t)\cdot e^{-3t}} + \cancel{6v'(t) e^{-3t}} \cancel{-18 v(t) e^{-3t}} + \cancel{9 v(t) e^{-3t}} = 0$.

$v''(t) = 0 \rightarrow v'(t) = c_1 \longrightarrow v(t) = c_1 t + c_2$

$y_0 = (c_1 t + c_2) e^{-3t} = c_1 \underbrace{t\, e^{-3t}}_{y_2} + c_2 \underbrace{e^{-3t}}_{y_1}$

Νομίζω μπορούμε να πούμε με τη μία $y_2 = t\cdot y_1 = t\cdot e^{-3t}$

αλλά αυτή είναι η απόδειξη

$$W = \begin{vmatrix} e^{-3t} & t\cdot e^{-3t} \\ -3e^{-3t} & e^{-3t} - 3te^{-3t} \end{vmatrix} = e^{-6t} \cancel{- 3te^{-6t}} + \cancel{3te^{-6t}} = e^{-6t} \neq 0$$

$g(t) = e^{-3t} + \cos 3t$

$$Y = -y_1(t)\cdot \int_{t_0}^{t} \frac{y_2(s)\, g(s)}{W}\, ds + y_2(t) \int_{t_0}^{t} \frac{y_1(s)\, g(s)}{W}\, ds$$

μπορούμε να βρούμε τη μερική λύση από αυτή τη σχέση αλλά το ολοκλήρωμα είναι λίγο δύσκολο οπότε τη βρίσκω

Δοκιμή $\bar{Y} = Ae^{-3t} + B\cos 3t + \Gamma \sin 3t$

$Y' = \dots$

$Y'' = \dots$

Αντικαθιστώ → δε βγαίνει

Δοκιμή $\bar{Y} = At \cdot e^{-3t} + B\cos 3t + \Gamma \sin 3t$

$Y' = \dots \qquad \bar{Y}'' = \dots$

Αντικαθιστώ → δε βγαίνει

Δοκιμή $\bar{Y} = At^2 \cdot e^{-3t} + B\cos 3t + \Gamma \sin 3t$

$$\bar{Y}' = 2Ate^{-3t} - 3At^2e^{-3t} - 3B\sin 3t + 3\Gamma\cos 3t$$

$$\bar{Y}'' = 2Ae^{-3t} - 6Ate^{-3t} - 6Ate^{-3t} + 9At^2e^{-3t} - 9B\cos 3t - 9\Gamma\sin 3t$$

$$2Ae^{-3t} - \cancel{12Ate^{-3t}} + \cancel{9At^2e^{-3t}} - \cancel{9B\cos 3t} - \cancel{9\Gamma\sin 3t}$$
$$+ \cancel{12Ate^{-3t}} - \cancel{18At^2e^{-3t}} - 18B\sin 3t + 18\Gamma\cos 3t$$
$$+ \cancel{9At^2e^{-3t}} + \cancel{9B\cos 3t} + \cancel{9\Gamma\sin 3t} = e^{-3t} + \cos 3t$$

$$2Ae^{-3t} - 18B\sin 3t + 18\Gamma\cos 3t = e^{-3t} + \cos 3t$$

$2A = 1 \rightarrow A = 1/2 \qquad -18B = 0 \rightarrow B = 0 \qquad 18\Gamma = 1 \Rightarrow \Gamma = \frac{1}{18}$

$$\underline{Y} = \frac{1}{2}e^{-3t} + \frac{1}{18}\sin 3t$$

$$y = y_0 + \bar{Y} = c_1te^{-3t} + c_2e^{-3t} + \frac{1}{2}t^2e^{-3t} + \frac{1}{18}\sin 3t$$

$$y = c_1 t \cdot e^{-3t} + c_2 \cdot e^{-3t} + \frac{1}{2}t^2 \cdot e^{-3t} + \frac{1}{18}\sin 3t$$

Θεμα 3

a) $y'' + 4y = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & \pi \le t < \infty \end{cases}$ $\quad y(0) = 1 \quad y'(0) = 0$

$$g(t) = \sin t - U_\pi(t)\cdot \sin t$$

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\sin t\} - \mathcal{L}\{U_\pi(t)\sin t\}$$

$$s^2 \mathcal{L}\{y\} - s\cdot y(0) - y'(0) + 4\mathcal{L}\{y\} = \frac{1}{s^2+1} - \mathcal{L}\{U_\pi(-\sin(t-\pi))\}$$

$$(s^2+4)\mathcal{L}\{y\} - s = \frac{1}{s^2+1} - e^{-\pi t}\cdot \mathcal{L}\{-\sin t\} \Rightarrow$$

$$(s^2+4)\mathcal{L}\{y\} = \frac{s^3+s+1}{s^2+1} - e^{-\pi t}\left(-\frac{1}{s^2+1}\right) \Rightarrow$$

$$\mathcal{L}\{y\} = \frac{s^3+s+1+e^{-\pi t}}{(s^2+1)(s^2+4)} = \frac{s^3+s+1}{(s^2+1)(s^2+4)} + e^{-\pi t}\frac{1}{(s^2+1)(s^2+4)}$$

$$\frac{s^2+2}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{\Gamma s + \Delta}{s^2+4} \Rightarrow$$

$$As^3 + 4As + Bs^2 + 4B + \Gamma s^3 + \Gamma s + \Delta s^2 + \Delta = s^3 + s + 1$$

$$s^3(A+\Gamma) + s^2(B+\Delta) + s(4A+\Gamma) + (4B+\Delta) = s^3 + s + 1$$

$A + \Gamma = 1 \rightarrow \Gamma = -A + 1$

$B + \Delta = 0 \rightarrow \Delta = -B$

$4A + \Gamma = 1 \rightarrow 3A = 0 \rightarrow A = 0 \rightarrow \Gamma = 1$

$4B + \Delta = 1 \rightarrow 3B = 1 \rightarrow B = 1/3 \quad \Delta = -1/3$

$A' + \Gamma' = 0 \rightarrow A = -\Gamma \rightarrow A = 0$

$B' + \Delta' = 0 \rightarrow B = -\Delta \rightarrow \Delta = -1/3$

$4A' + \Gamma' = 0 \rightarrow \Gamma = 0$

$4B' + \Delta' = 1 \rightarrow B = 1/3$

$$\mathcal{L}\{y\} = \left(\frac{1/3}{s^2+1} + \frac{s - 1/3}{s^2+4}\right) + e^{-\pi t}\cdot\left(\frac{1/3}{s^2+1} + \frac{-1/3}{s^2+4}\right)$$

$$\mathcal{L}\{y\} = \frac{1}{3}\mathcal{L}\{\sin t\} + \mathcal{L}\{\cos 2t\} - \frac{1}{3}\mathcal{L}\{\sin t\} + \frac{1}{3}e^{-\pi t}\mathcal{L}\{\sin t\} - \frac{1}{3}e^{-\pi t}\mathcal{L}\{\sin 2t\}$$

$$\mathcal{L}\{y\} = \frac{1}{3}\mathcal{L}\{\sin t\} + \mathcal{L}\{\cos 2t\} - \frac{1}{3}\mathcal{L}\{\sin t\} + \frac{1}{3}\mathcal{L}\{u_\pi(t)\cdot \sin(t-\pi)\} - \frac{1}{3}\mathcal{L}\{u_\pi(t)\sin[2(t-\pi)]\}$$

$$y = \frac{1}{3}\sin t + \cos 2t - \frac{1}{3}\sin t + \frac{1}{3}u_\pi(t)\sin(t-\pi) - \frac{1}{3}u_\pi(t)\sin(2(t-\pi))$$

B) $x'(t) = \begin{pmatrix} -9 & 5 \\ -7 & 3 \end{pmatrix} x(t) \qquad x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$x' = A x$

$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} -9-\lambda & 5 \\ -7 & 3-\lambda \end{vmatrix} = 0 \Rightarrow$

$(-9-\lambda)(3-\lambda) + 35 = 0 \Rightarrow -27 + 9\lambda - 3\lambda + \lambda^2 + 35 = 0 \Rightarrow$

$\lambda^2 + 6\lambda + 8 = 0 \Rightarrow (\lambda+4)(\lambda+2) = 0 \Rightarrow \lambda = -4$ ή $\lambda = -2$

Για $\lambda = -2$: $\begin{pmatrix} -7 & 5 \\ -7 & 5 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \Rightarrow -7\xi_1 + 5\xi_2 = 0 \Rightarrow$

$\xi_2 = \frac{7}{5}\xi_1$

$\xi^{(1)} = \begin{pmatrix} 1 \\ \frac{7}{5} \end{pmatrix} \qquad x^{(1)} = \begin{pmatrix} 1 \\ \frac{7}{5} \end{pmatrix} \cdot e^{-2t}$

Για $\lambda = -4$: $\begin{pmatrix} 5 & 5 \\ -7 & 7 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \Rightarrow \xi_1 = \xi_2$

$\xi^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad x^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-4t}$

$x = c_1 \cdot \begin{pmatrix} 1 \\ \frac{7}{5} \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-4t}$

$x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow c_1 \cdot \begin{pmatrix} 1 \\ 7/5 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow$

$c_1 + c_2 = 1 \rightarrow c_2 = 1 - c_1$

$\frac{7}{5}c_1 + c_2 = 1 \rightarrow c_1 = 0$

$c_2 = 1$

$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-4t}$