Lecture Note for IISEE

Fundamentals of Structural Dynamics

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This lecture note was originally written by Dr. Makoto Watabe and Dr. Yuji Ishiyama for the participants of the International Institute of Seismology and Earthquake Engineering (IISEE), Building Research Institute (BRI). After that, Dr. Izuru Okawa revised the note, adding descriptions, examples how to solve questions, etc. Then Prof. Yuji Ishiyama who moved from BRI to Hokkaido University further revised the note for the graduate students of English Graduate Program for Socio-Enviromental Engineering (EGPSEE), Graduate School of Engineering, Hokkaido University.

The authors do not intend that the note be used for professional engineers or highly educated researchers, but for those who have just started learning structural engineering. Therefore the note only contains the fundamental concepts in structural dynamics. The authors hope that readers will look into professional books to understand the background in more detail.

If you have questions, suggestions, or comments on this lecture note, please write to us. We thank you in advance for taking the time and interest to do so.

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Contents

Chapter 1

Introduction

What is the difference between dynamic loading and static loading?

Now, imagine you are standing in a train at rest. The train then begins to raise its speed from 0 to 100 kilometers per hour (km/h). In the train, you will probably incline your body forward so as not to fall down to the floor. On the other hand, when the train reduces its speed on reaching a station, you will incline yourself in the opposite way. You know this through experience. How can this phenomenon be explained in physical terms? In addition, you need a greater angle of inclination when the speed is changed more rapidly. Why is this additional inclination necessary?

You will easily find many good examples such as this issue associated with problems in dynamics in daily life. However, you will probably have some difficulty in explaining the theoretical background of such problems.

Most of you are familiar with statics, in which the concept of time is not involved. Hooke's law tells us that deformation is proportional to applied force. This is true. In the real world, however, applying a load without accompanying lapse of time is impossible. Therefore, we frequently encounter cases in which we must take into consideration how rapidly the action (loading) is applied. Here, we must consider a force other than the static force. This is the force associated with time, i.e. the dynamic force.

Let us go back to the "train" example. When the above mentioned phenomenon

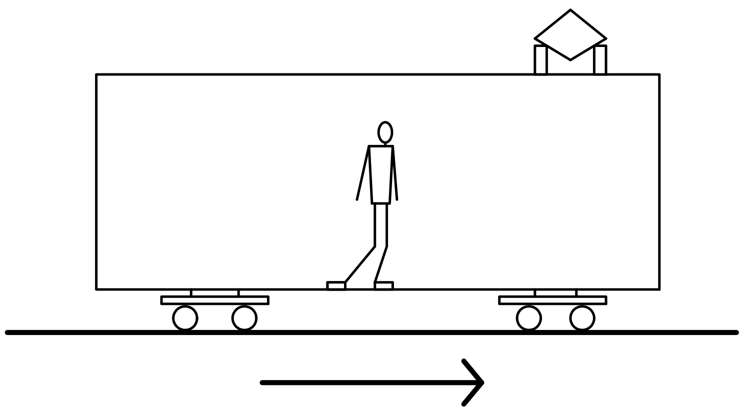


Fig.1.1 Human action in a train

is considered theoretically, the issue of the change in speed, i.e. acceleration, arises. (We always feel the acceleration of gravity toward the center of the earth, as you know. Therefore, we feel our own weight.) In statics, we assume that the loading is done infinitely slowly. The induced acceleration is so small that it can be neglected.

However, when the loading is applied quickly enough, the inertia force grows large enough to be comparable to the other forces so it cannot be neglected in the equation of force equilibrium. Therefore, the inertia force, which is caused by acceleration, must be taken into account in dynamics. There are several more things to be considered in dynamics, such as the damping force, etc.

This lecture note covers how to estimate the dynamic behavior of single degree of freedom (SDOF) systems and multi-degree of freedom (MDOF) systems. The analytical methods are mainly concerned with the deterministic and time domain procedures. The modal analysis, that involves the response spectrum method and the square root of sum of squares (SRSS) rule, is also included to estimate the maximum response of a structure in a stochastic manner. In addition, the structural system properties are mainly considered to be linearly elastic, and it is assumed that they do not change with respect to time. The last chapter, however, briefly deals with nonlinear analysis.

Chapter 2

Single Degree of Freedom (SDOF) Systems

Let us start from the simplest case. A building is idealized as shown in Fig.2.1. You may not imagine that there can actually be a structure so simple that it consists of a bar with a ball-like weight on the top or an assembly of mass, dashpot and spring. This is an imaginary model in which the mass is allowed to move in only one direction. Therefore, it is called a single degree of freedom (SDOF) system. We will discuss the behavior of this simplest model at first.

We know that the weight on the moon differs from the weight on the earth. This is because the acceleration due to gravity for the two is not the same. The weight changes but the mass that is proportional to the weight does not change. The force caused by the acceleration and mass is called the inertia force.

The damping force is related to velocity. Imagine a movable piston fixed into a cylinder filled with some liquid inside. When you move the piston, you feel resistance. The quicker you pull or push, the greater is the resistance force. This is a good example of viscous damping.

Another force is caused by a spring when it deforms. This force is sometimes called restoring force or elastic resistance force.

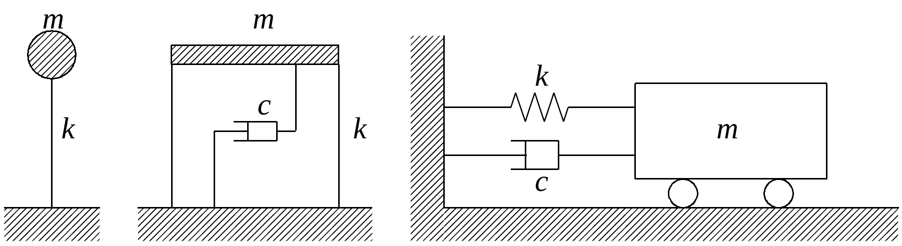


Fig.2.1 Analytical models of single degree of freedom (SDOF) systems

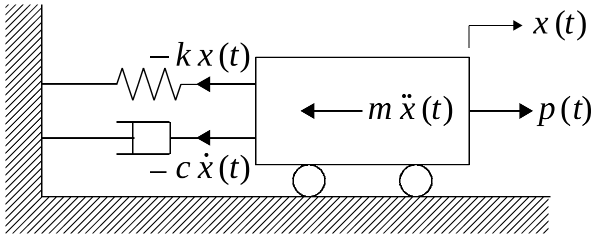


Fig.2.1.1 D'Alembert's principle considering dynamic equilibrium

2.1 Equation of Motion

The equation of motion is most important because structural response is computed as the solution to the equation. The equation of motion of a SDOF system can be given using d'Alembert's principle while considering the dynamic equilibrium (see Fig.2.1.1).

$$-m\,\ddot{x}(t) - c\,\dot{x}(t) - k\,x(t) + p(t) = 0$$

where m, c, k and $x(t)$ represent mass, damping, stiffness and displacement of the system, respectively, the upper dots represent differentiation with respect to time, and $p(t)$ denotes the applied external force.

The above equation can be written in the form

$$m\,\ddot{x}(t) + c\,\dot{x}(t) + k\,x(t) = p(t) \tag{2.1.1}$$

where the first, second and third terms of the left hand side are called the inertia force, damping force and elastic force, respectively.

2.2 Free Vibration

If we let the right hand side of Eq.(2.1.1) be equal to zero, the equation of motion without any applied forces, i.e. for free vibration, can be obtained as follows:

$$m\,\ddot{x}(t) + c\,\dot{x}(t) + k\,x(t) = 0 \tag{2.2.1}$$

The solution of the above equation is conventionally given as follows:

$$\begin{aligned} x(t) &= D\,e^{st} \\ \dot{x}(t) &= sD\,e^{st} \\ \ddot{x}(t) &= s^2 D\,e^{st} \end{aligned} \tag{2.2.2}$$

where D is an arbitrary constant. Substituting these equations into Eq.(2.2.1), we get

$$(m\,s^2 + c\,s + k)D\,e^{st} = 0$$

The above equation must be always satisfied. $D\,e^{st}$ changes with time, so that the value inside the parentheses should be equal to zero.

$$m\,s^2 + c\,s + k = 0 \tag{2.2.3}$$

Then, the roots of this equation are

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \tag{2.2.4}$$

Therefore, the solution for Eq.(2.2.1) is of the form

$$x(t) = D_1\, e^{s_1 t} + D_2\, e^{s_2 t} \tag{2.2.5a}$$

In case the two roots are equal, i.e. $s_1 = s_2 = s$, we have

$$\left(\frac{c}{2m}\right)^2 = \frac{k}{m}$$

Then the solution for Eq.(2.2.1) is

$$x(t) = (D_1 + D_2\, t)\, e^{st} \tag{2.2.5b}$$

where D_1 and D_2 are arbitrary constants.

The application of Eqs.(2.2.5a) and (2.2.5b) will be discussed in the following sections.

(1) Undamped Free Vibration

If the system is undamped, i.e. the damping coefficient c = 0, and let us introduce the notation

$$\omega_{\mathrm{n}}^2 = \frac{k}{m} \tag{2.2.6}$$

We will understand that ω_{n} is the undamped natural circular frequency [see Eq.(2.2.10)], and Eq.(2.2.4) becomes

$$s_{1,2} = \pm i\, \omega_{\mathrm{n}} \tag{2.2.7}$$

Then Eq.(2.2.5a) will be

$$x(t) = D_1\, e^{i\,\omega_{\mathrm{n}} t} + D_2\, e^{-i\,\omega_{\mathrm{n}} t} \tag{2.2.8}$$

By introducing Euler's formula, i.e.

$$e^{\pm i\,\omega_{\mathrm{n}} t} = \cos\omega_{\mathrm{n}} t \pm i \sin\omega_{\mathrm{n}} t \tag{2.2.9}$$

and noting that D_1 and D_2 can be complex, let

$$\begin{aligned} D_1 &= a + i\, b \\ D_2 &= c + i\, d \end{aligned}$$

Then,

$$\begin{aligned} x(t) &= (a + i\, b)(\cos\omega_{\mathrm{n}} t + i \sin\omega_{\mathrm{n}} t) + (c + i\, d)(\cos\omega_{\mathrm{n}} t - i \sin\omega_{\mathrm{n}} t) \\ &= \{(a + c) + i(b + d)\} \cos\omega_{\mathrm{n}} t + \{(d - b) + i(a - c)\} \sin\omega_{\mathrm{n}} t \end{aligned}$$

Here, if we let

$$\begin{aligned} A &= (a + c) + i(b + d) \\ B &= (d - b) + i(a - c) \end{aligned}$$

Thus, Eq.(2.2.8) becomes

$$x(t) = A\cos\omega_{\mathrm{n}}t + B\sin\omega_{\mathrm{n}}t \tag{2.2.10}$$

Note that $x(t)$ is real. Therefore, the imaginary parts of A and B should be zero. Then,

$$\begin{aligned} c &= a \\ d &= -b \end{aligned}$$

Therefore, the constant D_2 is the complex conjugate of D_1. A and B will be determined by the initial conditions. For example, in case $x(0)$ and $\dot{x}(0)$ are already given, by substituting these into Eq.(2.2.10) and into its derivative

$$\dot{x}(t) = -\omega_{\mathrm{n}}A\sin\omega_{\mathrm{n}}t + \omega_{\mathrm{n}}B\cos\omega_{\mathrm{n}}t \tag{2.2.11}$$

we obtain

$$\begin{aligned} x(0) &= A \\ \dot{x}(0) &= \omega_{\mathrm{n}}B \end{aligned}$$

Thus, Eq.(2.2.10) becomes

$$x(t) = x(0)\cos\omega_{\mathrm{n}}t + \frac{\dot{x}(0)}{\omega_{\mathrm{n}}}\sin\omega_{\mathrm{n}}t \tag{2.2.12}$$

Remembering the following formula

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

and referring to the right triangle in Fig.2.2.1, Eq.(2.2.12) can be written in the form

$$\begin{aligned} x(t) &= \rho\left\{\frac{x(0)}{\rho}\cos\omega_{\mathrm{n}}t + \frac{\dot{x}(0)/\omega_{\mathrm{n}}}{\rho}\sin\omega_{\mathrm{n}}t\right\} \\ &= \rho\{\cos\theta\cos\omega_{\mathrm{n}}t + \sin\theta\sin\omega_{\mathrm{n}}t\} \\ &= \rho\cos(\omega_{\mathrm{n}}t - \theta) \end{aligned} \tag{2.2.13}$$

where,

$$\rho = \sqrt{\{x(0)\}^2 + \left\{\frac{\dot{x}(0)}{\omega_{\mathrm{n}}}\right\}^2} \tag{2.2.14a}$$

$$\tan\theta = \frac{\dot{x}(0)}{\omega_{\mathrm{n}}x(0)} \tag{2.2.14b}$$

Eq.(2.2.13) represents a simple harmonic motion (SHM) and is shown in Fig.2.2.2. It is evident that ω_{n} is the circular frequency or angular velocity of the motion for undamped systems. The cyclic frequency f_{n}, which is frequently referred to merely as the frequency of the motion, is given by

$$f_{\mathrm{n}} = \frac{\omega_{\mathrm{n}}}{2\pi} \tag{2.2.15a}$$

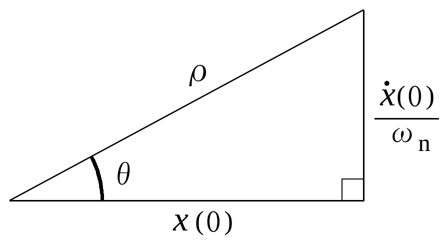


Fig.2.2.1 Sides and hypotenuse of the right triangle

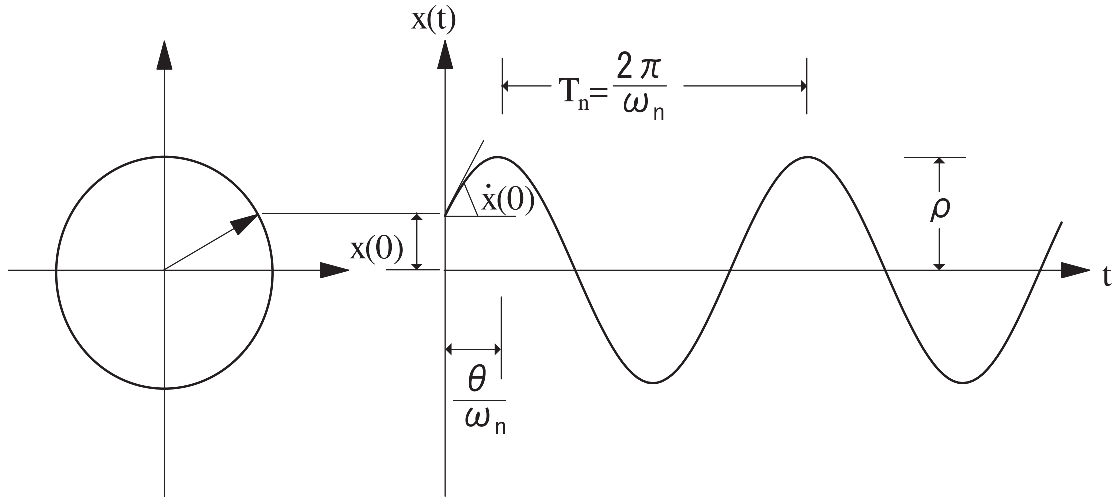


Fig.2.2.2 Undamped free-vibration response

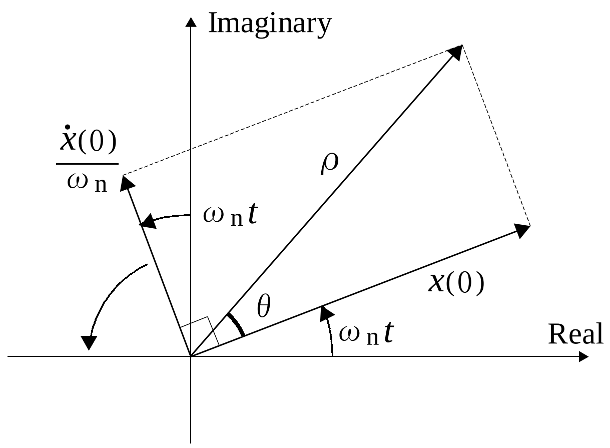


Fig.2.2.3 Argand diagram

The reciprocal of the frequency is called the period T_{n}.

$$T_{\mathrm{n}} = \frac{1}{f} = \frac{2\pi}{\omega_{\mathrm{n}}} \tag{2.2.15b}$$

Eq.(2.2.13) is also shown in the Argand diagram of Fig.2.2.3. The response $x(t)$ is given by the real part of the rotating vector and $\dot{x}(t)/\omega_{\mathrm{n}}$ by the imaginary part.

[**Example 2.1**]

Set up the equation of motion for the following undamped SDOF systems without external forces and solve for the response under the given conditions.

a)

$$m = 20 \text{ ton}, \quad k = 180 \text{ kN/cm}$$
$$x(0) = 1 \text{ cm}, \quad \dot{x}(0) = -3 \text{ cm/s}$$

b)

$$w = 9,800 \text{ kN}, \quad k = 250 \text{ kN/cm}$$
$$x(0) = 5 \text{ cm}, \quad \dot{x}(\pi) = 5 \text{ cm/s}$$

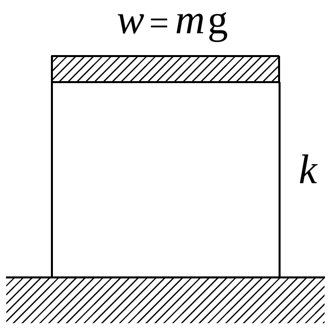


Fig.E2.1

[**Solution**]

a) The equation of motion is

$$20\ddot{x}(t) + 180x(t) = 0$$

$$\omega_{\mathrm{n}} = \sqrt{\frac{k}{m}} = \sqrt{\frac{180}{20}} = 3 \text{ (rad/s)}$$

Then,

$$x(t) = A\cos\omega_{\mathrm{n}}t + B\sin\omega_{\mathrm{n}}t = A\cos 3t + B\sin 3t$$
$$\dot{x}(t) = -3A\sin 3t + 3B\cos 3t$$

Substituting the above conditions $x(0) = 1$, $\dot{x}(0) = -3$

$$1 = A\cos 0 + B\sin 0 = A$$
$$-3 = -3A\sin 0 + 3B\cos 0 = 3B$$

Therefore,

$$A = 1, \ B = -1$$

That is,

$$x(t) = \cos 3t - \sin 3t$$

b) At first, the mass is computed as w/g, where g is the acceleration due to gravity, i.e., 980 cm/s^2. Then $m = 10$ The equation of motion is,

$$10\ddot{x}(t) + 250x(t) = 0$$

$$\omega_{\rm n} = \sqrt{\frac{k}{m}} = \sqrt{\frac{250}{10}} = 5$$

Then,

$$\begin{aligned} x(t) &= A\cos\omega_{\rm n}t + B\sin\omega_{\rm n}t = A\cos 5t + B\sin 5t \\ \dot{x}(t) &= -5A\sin 5t + 5B\cos 5t \end{aligned}$$

Substituting the above conditions $x(0) = 5$, $\dot{x}(\pi) = 5$

$$\begin{aligned} 5 &= A\cos 0 + B\sin 0 = A \\ 5 &= -5A\sin 5\pi + 5B\cos 5\pi = -5B \end{aligned}$$

Therefore,

$$A = 5,\ B = -1$$

That is,

$$x(t) = 5\cos 5t - \sin 5t$$

(2) Damped Free Vibration

If a damping force exists, the solution is classified into three cases according to whether the value under the square root sign in Eq.(2.2.4) is positive, negative or zero.

i) Critical Damping

In the special case where the radical in Eq.(2.2.4) vanishes, we have

$$\left(\frac{c}{2\,m}\right)^2 = \frac{k}{m} = \omega_{\rm n}^2 \tag{2.2.16a}$$

The latter equality is given by Eq.(2.2.6). This is the condition of critical damping and the critical damping coefficient $c_{\rm cr}$ is

$$c_{\rm cr} = 2\,m\,\omega_{\rm n} \tag{2.2.16b}$$

The solution of Eq.(2.2.3) in this case is as was already given by Eq.(2.2.5b)

$$x(t) = (D_1 + D_2\,t)\,e^{st}$$

where,

$$s = -\frac{c_{\rm cr}}{2m} = -\omega_{\rm n}$$

Then,

$$x(t) = (D_1 + D_2\,t)\,e^{-\omega_{\rm n}t} \tag{2.2.17}$$

Introducing initial conditions,

$$x(t) = \{x(0)(1 + \omega_{\rm n}t) + \dot{x}(0)\,t\}\,e^{-\omega_{\rm n}t} \tag{2.2.18}$$

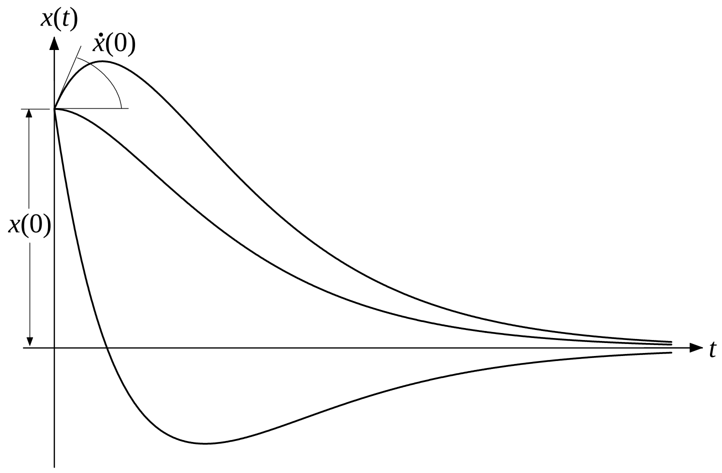


Fig.2.2.4 Free vibration response of a critically damped system

This is shown in Fig.2.2.4. We can see that the free vibration response of a critically damped system does not oscillate, but rather returns to zero displacement, because of the exponential factor in Eq.(2.2.18).

ii) Underdamped System

If the damping is less than critical, we have, from Eq.(2.2.16b)

$$c < 2\,m\,\omega_\mathrm{n}$$

and the radical in Eq.(2.2.4) becomes negative. In this case, it is convenient to express the damping as a ratio of the critical damping value c_cr. Thus,

$$\zeta = \frac{c}{c_\mathrm{cr}} = \frac{c}{2\,m\,\omega_\mathrm{n}} < 1 \tag{2.2.19}$$

where ζ is called the damping ratio or the fraction of critical damping. Substituting this into Eq.(2.2.4) yields

$$\begin{aligned} s_{1,2} &= -\zeta\omega_\mathrm{n} \pm \sqrt{(\zeta\omega_\mathrm{n})^2 - \omega_\mathrm{n}^2} \\ &= -\zeta\omega_\mathrm{n} \pm i\,\omega_\mathrm{d} \end{aligned} \tag{2.2.20}$$

where,

$$\omega_\mathrm{d} = \omega_\mathrm{n}\sqrt{1-\zeta^2} \tag{2.2.21}$$

The value ω_d is called the damped natural circular frequency. Generally, it differs very little from the undamped natural circular frequency ω_n, because the damping ratios ζ in typical structural systems are not very large ($\zeta < 0.2$). The relationship between the ratio $\omega_\mathrm{d}/\omega_\mathrm{n}$ and ζ is indicated by a circle, as illustrated in Fig 2.2.5. The figure shows that $\omega_\mathrm{d}/\omega_\mathrm{n}$ is close to unity when $\zeta < 0.2$.

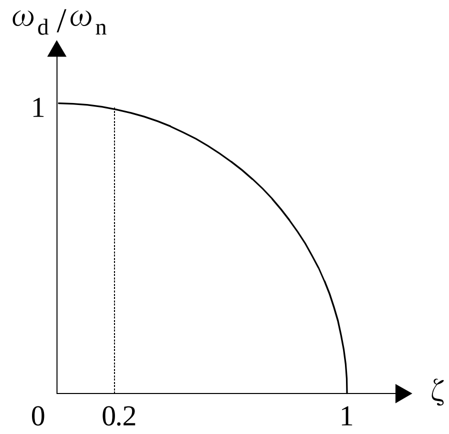


Fig.2.2.5 Relationship between the ratio $\omega_\mathrm{d}/\omega_\mathrm{n}$ and ζ

The free vibration response of an underdamped system is given by substituting Eq.(2.2.20) into Eq.(2.2.5a)

$$\begin{aligned} x(t) &= D_1\, e^{(-\zeta\omega_\mathrm{n}+i\,\omega_\mathrm{d})t} + D_2\, e^{(-\zeta\omega_\mathrm{n}-i\,\omega_\mathrm{d})t} \\ &= e^{-\zeta\omega_\mathrm{n}t}(D_1\, e^{i\,\omega_\mathrm{d}t} + D_2\, e^{-i\,\omega_\mathrm{d}t}) \\ &= e^{-\zeta\omega_\mathrm{n}t}(A\cos\omega_\mathrm{d}t + B\sin\omega_\mathrm{d}t) \end{aligned} \tag{2.2.22a}$$

Introducing initial conditions, we have,

$$x(t) = e^{-\zeta\omega_\mathrm{n}t}\left\{x(0)\cos\omega_\mathrm{d}t + \frac{\dot{x}(0)+\zeta\omega_\mathrm{n}x(0)}{\omega_\mathrm{d}}\sin\omega_\mathrm{d}t\right\} \tag{2.2.22b}$$

This above expression can be written in rotating-vector form.

$$x(t) = e^{-\zeta\omega_\mathrm{n}t}\rho\cos(\omega_\mathrm{d}t-\theta) \tag{2.2.23}$$

where,

$$\rho = \sqrt{\{x(0)\}^2 + \left\{\frac{\dot{x}(0)+\zeta\omega_\mathrm{n}x(0)}{\omega_\mathrm{d}}\right\}^2} \tag{2.2.24}$$

$$\tan\theta = \frac{\{\dot{x}(0)+\zeta\omega_\mathrm{n}x(0)\}/\omega_\mathrm{d}}{x(0)} \tag{2.2.25}$$

The free vibration response of an underdamped system is illustrated in Fig.2.2.6. It is noted that the system oscillates about a neutral position (the zero base line) with a constant circular frequency ω_d or with a damped period $T_\mathrm{d} = 2\pi/\omega_\mathrm{d}$. The rotating vector representation is the same as shown in Fig.2.2.3 except that the length of the vector diminishes exponentially.

Let us consider any two successive positive peaks shown in Fig.2.2.6, i.e. x_i and x_{i+1}. If we assume from Eq.(2.2.23)

$$x_i = \rho\, e^{-\zeta\omega_\mathrm{n}t_i}$$

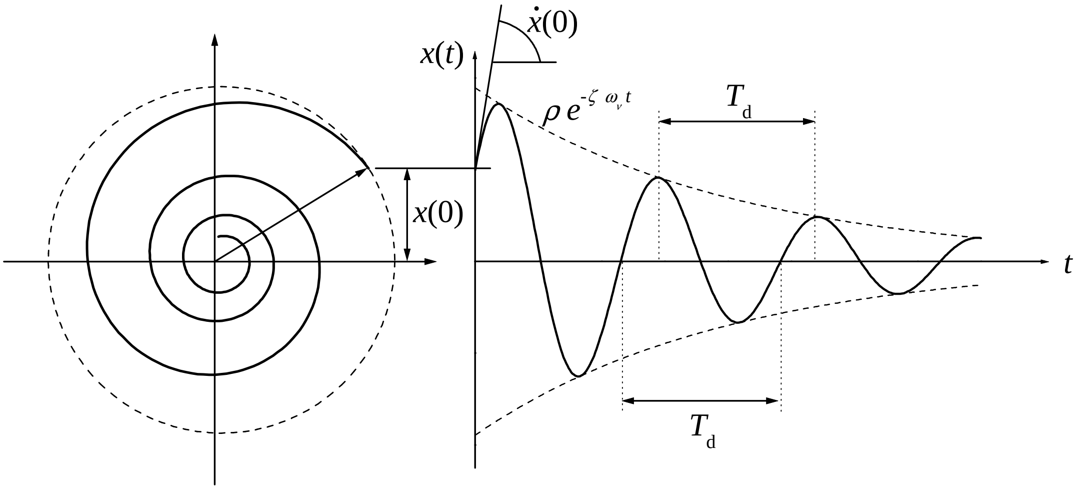


Fig.2.2.6 Free vibration response of an underdamped system

then,

$$x_{i+1} = \rho\, e^{-\zeta\omega_\mathrm{n}(t_i + \frac{2\pi}{\omega_\mathrm{d}})}$$

The ratio of the above two quantities is

$$\frac{x_i}{x_{i+1}} = e^{\zeta\omega_\mathrm{n}\frac{2\pi}{\omega_\mathrm{d}}} \tag{2.2.26a}$$

Hence,

$$\ln\frac{x_i}{x_{i+1}} = 2\pi\zeta\frac{\omega_\mathrm{n}}{\omega_\mathrm{d}} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{2.2.26b}$$

For low damping, the above equation becomes

$$\ln\frac{x_i}{x_{i+1}} \approx 2\pi\zeta \tag{2.2.27}$$

If we consider the Taylor series expansion, *

$$\frac{x_i}{x_{i+1}} = e^{2\pi\zeta} = 1 + 2\pi\zeta + \frac{(2\pi\zeta)^2}{2!} + \dots$$

Therefore, for low values of ζ, we have

$$\zeta \approx \frac{x_i - x_{i+1}}{2\pi x_{i+1}} \tag{2.2.28}$$

For lightly damped systems, a reliable estimate of the damping ratio can be obtained by considering peaks which are several cycles apart. Then,

$$\ln\frac{x_i}{x_{i+n}} = 2\pi n\zeta\frac{\omega_\mathrm{n}}{\omega_\mathrm{d}} \tag{2.2.29}$$

* $f(z) = f(a) + f'(a)(z-a) + f''(a)\frac{(z-a)^2}{2!} + f'''(a)\frac{(z-a)^3}{3!} + \cdots$

For very low damping,

$$\zeta \approx \frac{x_i - x_{i+n}}{2\pi n\, x_{i+n}} \tag{2.2.30}$$

iii) Overdamped system

If the damping is greater than critical, it is called overdamped, although this is not encountered in usual structural systems. In this case $\zeta > 1$, and Eq.(2.2.4) can be written

$$\begin{aligned} s_{1,2} &= -\zeta\omega_{\mathrm{n}} \pm \omega_{\mathrm{n}}\sqrt{\zeta^2 - 1} \\ &= -\zeta\omega_{\mathrm{n}} \pm \omega'_{\mathrm{n}} \end{aligned} \tag{2.2.31}$$

where

$$\omega'_{\mathrm{n}} = \omega_{\mathrm{n}}\sqrt{\zeta^2 - 1} \tag{2.2.32}$$

Substituting Eq.(2.2.31) into Eq.(2.2.5), we have

$$x(t) = e^{-\zeta\omega_{\mathrm{n}} t}(D_1\, e^{\omega'_{\mathrm{n}} t} + D_2\, e^{-\omega'_{\mathrm{n}} t}) \tag{2.2.33a}$$

If we remember the following two formulae,

$$\sinh\zeta = \frac{e^{\zeta} - e^{-\zeta}}{2}$$

$$\cosh\zeta = \frac{e^{\zeta} + e^{-\zeta}}{2}$$

Eq.(2.2.33a) becomes

$$x(t) = e^{-\zeta\omega_{\mathrm{n}} t}(A\cosh\omega'_{\mathrm{n}} t + B\sinh\omega'_{\mathrm{n}} t) \tag{2.2.33b}$$

This is very similar to the case of critical damping and is of no physical interest, but this system will oscillate if external forces are applied.

[**Example 2.2**]

Determine the response of the following system under the given initial conditions.

$$k = 320,\ c = 4,\ m = 5$$

$$x(0) = 1,\ \dot{x}(0) = 7.6$$

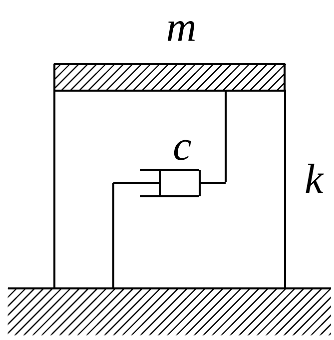


Fig.E2.2

[**Solution**]

The equation of motion of the system is

$$5\,\ddot{x}(t) + 4\,\dot{x}(t) + 320\,x(t) = 0$$

The characteristic equation is

$$5\,s^2 + 4\,s + 320 = 0$$

Then,

$$s = -0.4 \pm 7.99\,i \approx -0.4 \pm 8\,i$$

Therefore,

$$x(t) = e^{-0.4t}(A\cos 8t + B\sin 8t)$$

The first derivative of the above equation is

$$\dot{x}(t) = -0.4\,e^{-0.4t}(A\cos 8t + B\sin 8t) + e^{-0.4t}(-8A\sin 8t + 8B\cos 8t)$$

Substituting the initial conditions $x(0)$ and $\dot{x}(0)$, we have

$$A = 1,\ B = 1$$

Therefore,

$$x(t) = e^{-0.4t}(\cos 8t + \sin 8t)$$

or

$$x(t) = \sqrt{2}\,e^{-0.4t}\cos(8t - \frac{\pi}{4})$$

2.3 Response to Harmonic Loading

If the system in Fig.2.1.1 is subjected to a harmonically varying load of an amplitude p_0 and a circular frequency ω, the equation of motion is

$$m\,\ddot{x}(t) + c\,\dot{x}(t) + k\,x(t) = p_0 \sin\omega t \tag{2.3.1}$$

The general solution of the above equation is the sum of the complementary function $x_{\rm c}(t)$ and the particular integral $x_{\rm p}(t)$. Then,

$$x(t) = x_{\rm c}(t) + x_{\rm p}(t) \tag{2.3.2}$$

(1) Undamped Systems

If the system is undamped, the equation of motion becomes

$$m\,\ddot{x}(t) + k\,x(t) = p_0 \sin\omega t \tag{2.3.3}$$

The complementary function is the free vibration response of Eq.(2.3.3) and it is, as already given by Eq.(2.2.10),

$$x_{\rm c}(t) = A\cos\omega_{\rm n}t + B\sin\omega_{\rm n}t \tag{2.3.4}$$

The response to the harmonic loading can be assumed to be harmonic. Thus the particular integral is

$$x_{\rm p}(t) = G_1\cos\omega t + G_2\sin\omega t \tag{2.3.5}$$

Substituting the above equation and its derivative into Eq.(2.3.3), we have

$$G_1 = 0 \tag{2.3.6}$$

$$G_2 = \frac{p_0}{k}\frac{1}{1-r_{\mathrm{n}}^2} \tag{2.3.7}$$

where r_{n} is the frequency ratio given by

$$r_{\mathrm{n}} = \frac{\omega}{\omega_{\mathrm{n}}} \tag{2.3.8}$$

Therefore, the general solution of Eq.(2.3.3) is

$$x(t) = A\cos\omega_{\mathrm{n}}t + B\sin\omega_{\mathrm{n}}t + \frac{p_0}{k}\frac{1}{1-r_{\mathrm{n}}^2}\sin\omega t \tag{2.3.9}$$

$G_1 = 0$ indicates that the steady state response is in phase to harmonically applied load. This is because the damping is zero.

If the system is initially at rest, i.e. $x(0) = 0, \dot{x}(0) = 0$, the above equation becomes

$$x(t) = \frac{p_0}{k}\frac{1}{1-r_{\mathrm{n}}^2}(\sin\omega t - r_{\mathrm{n}}\sin\omega_{\mathrm{n}}t) \tag{2.3.10}$$

where p_0/k is called the static displacement, because this is the value which would be produced by the load p_0 applied statically, and $1/(1-r_{\mathrm{n}}^2)$ is called the dynamic magnification factor (R_{dy}), which represents the dynamic amplification effect of a harmonically applied load.

When we derived Eq.(2.3.9), we had assumed that $r_{\mathrm{n}} \neq 1$. But if $r_{\mathrm{n}} = 1$ or $\omega = \omega_{\mathrm{n}}$, the particular integral should have the form of

$$x_{\mathrm{p}}(t) = G_1\, t\cos\omega_{\mathrm{n}}t + G_2\, t\sin\omega_{\mathrm{n}}t \tag{2.3.11}$$

Substituting this and its derivative into Eq.(2.3.3), we have

$$G_1 = -\frac{p_0}{2m\omega_{\mathrm{n}}}$$

$$G_2 = 0$$

Then,

$$x_{\mathrm{p}}(t) = -\frac{p_0}{2m\omega_{\mathrm{n}}}\, t\cos\omega_{\mathrm{n}}t \tag{2.3.12}$$

Therefore, the general solution is

$$x(t) = A\cos\omega_{\mathrm{n}}t + B\sin\omega_{\mathrm{n}}t - \frac{p_0}{2m\omega_{\mathrm{n}}}\, t\cos\omega_{\mathrm{n}}t \tag{2.3.13}$$

Substituting the condition that the system is initially at rest (this means that the velocity and displacement at $t = 0$ are both zero), we have

$$\begin{aligned} x(t) &= \frac{p_0}{2m\omega_{\mathrm{n}}^2}(\sin\omega_{\mathrm{n}}t - \omega_{\mathrm{n}}t\cos\omega_{\mathrm{n}}t) \\ &= \frac{p_0}{2k}(\sin\omega_{\mathrm{n}}t - \omega_{\mathrm{n}}t\cos\omega_{\mathrm{n}}t) \end{aligned} \tag{2.3.14}$$

since

$$\omega_{\mathrm{n}}^2 = \frac{k}{m}$$

Some examples of Eqs.(2.3.10) and (2.3.14) are shown in Fig.2.3.1 as a parameter of $r_{\mathrm{n}} = \omega/\omega_{\mathrm{n}}$.

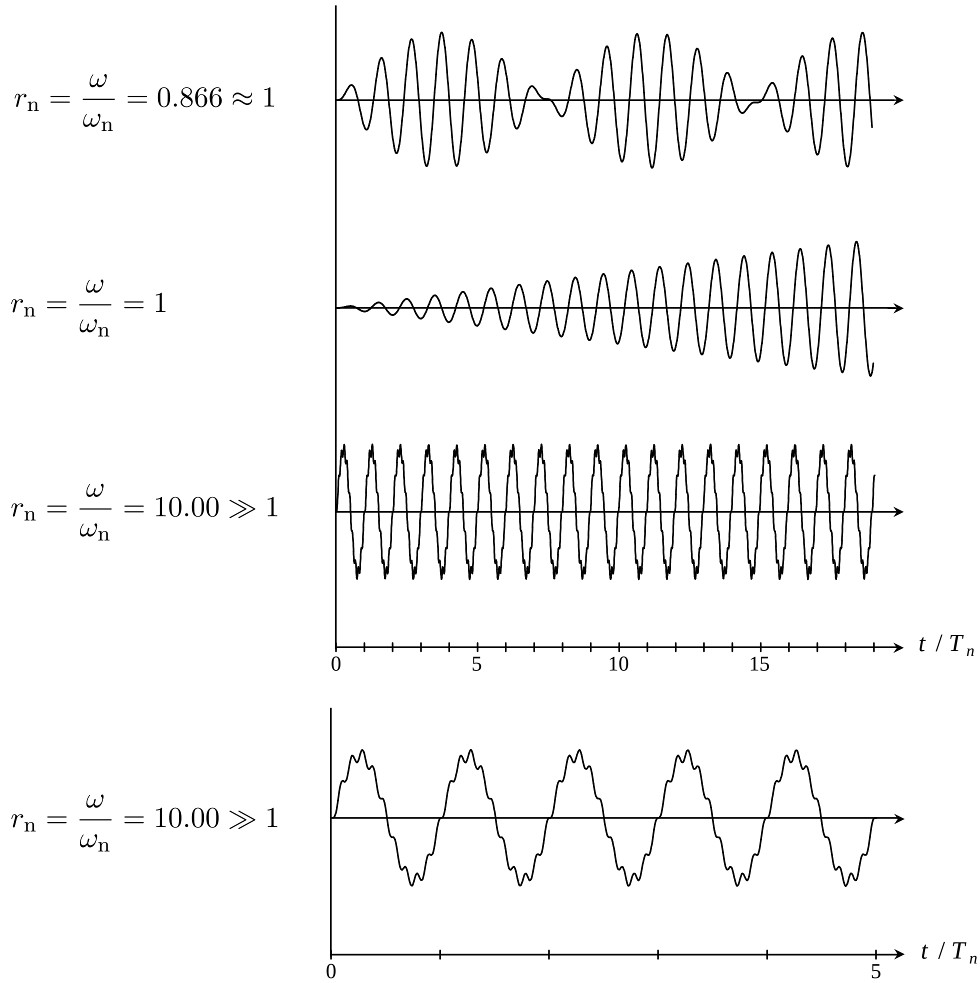


Fig.2.3.1 Response of undamped systems to harmonic loading

(2) Underdamped Systems

If the system is underdamped, the complementary function $x_{\mathrm{c}}(t)$ and the particular integral $x_{\mathrm{p}}(t)$ are given by

$$x_{\mathrm{c}}(t) = e^{-\zeta\omega_{\mathrm{n}}t}(A\cos\omega_{\mathrm{d}}t + B\sin\omega_{\mathrm{d}}t) \tag{2.3.15}$$

$$x_{\mathrm{p}}(t) = G_1\cos\omega t + G_2\sin\omega t \tag{2.3.16}$$

Substituting Eq.(2.3.16) and its derivatives into Eq.(2.3.1) yields

$$(-\omega^2 G_1 + \frac{c}{m}\omega\, G_2 + \frac{k}{m}G_1)\cos\omega t + (-\omega^2 G_2 - \frac{c}{m}\omega\, G_1 + \frac{k}{m}G_2 - \frac{p_0}{m})\sin\omega t = 0$$

From Eqs. (2.2.6) and (2.2.19), $\dfrac{k}{m} = \omega_{\mathrm{n}}^2$ and $\dfrac{c}{m} = 2\zeta\omega_{\mathrm{n}}$, then we have

$$(1 - r_{\mathrm{n}}^2)G_1 + 2\zeta r_{\mathrm{n}} G_2 = 0 \tag{2.3.17a}$$

$$-2\zeta r_{\mathrm{n}} G_1 + (1 - \zeta^2)G_2 = \frac{p_0}{k} \tag{2.3.17b}$$

where r_{n} is the frequency ratio $(= \dfrac{\omega}{\omega_{\mathrm{n}}})$ as already given by Eq.(2.3.8). Therefore,

$$G_1 = \frac{-2\zeta r_{\mathrm{n}}}{(1 - r_{\mathrm{n}}^2)^2 + (2\zeta r_{\mathrm{n}})^2}\frac{p_0}{k} \tag{2.3.18a}$$

$$G_2 = \frac{1 - r_{\mathrm{n}}^2}{(1 - r_{\mathrm{n}}^2)^2 + (2\zeta r_{\mathrm{n}})^2}\frac{p_0}{k} \tag{2.3.18b}$$

Then, the general solution $x(t)$ can be given by

$$\begin{aligned} x(t) =& e^{-\zeta\omega_{\mathrm{n}} t}(A\cos\omega_{\mathrm{d}} t + B\sin\omega_{\mathrm{d}} t) \\ &+ \frac{p_0}{k}\frac{1}{(1 - r_{\mathrm{n}}^2)^2 + (2\zeta r_{\mathrm{n}})^2}\{(1 - r_{\mathrm{n}}^2)\sin\omega t - 2\zeta r_{\mathrm{n}}\cos\omega t\} \end{aligned} \tag{2.3.19}$$

where the first and the second terms are called the transient response and steady state response, respectively. This is because the first term will decrease exponentially and vanish due to the damping of the system, and only the second term will remain as a steady state response. The steady response is written in the form

$$x_{\mathrm{p}}(t) = \frac{p_0}{k}\frac{1}{\sqrt{(1 - r_{\mathrm{n}}^2)^2 + (2\zeta r_{\mathrm{n}})^2}}\sin(\omega t - \theta) \tag{2.3.20}$$

where the factor p_0/k is the static displacement as was referred to previously, and the second factor is the dynamic magnification factor R_{dy}

$$R_{\mathrm{dy}} = \frac{1}{\sqrt{(1 - r_{\mathrm{n}}^2)^2 + (2\zeta r_{\mathrm{n}})^2}} \tag{2.3.21}$$

There is a time shift between the input and the response due to damping and the phase angle is

$$\tan\theta = \frac{2\zeta r_{\mathrm{n}}}{1 - r_{\mathrm{n}}^2} \tag{2.3.22}$$

Figs. 2.3.2 and 2.3.3 show the dynamic magnification factor R_{dy} and the phase angle θ, respectively. It should be noted that both R_{dy} and θ are functions of ζ and r_{n}.

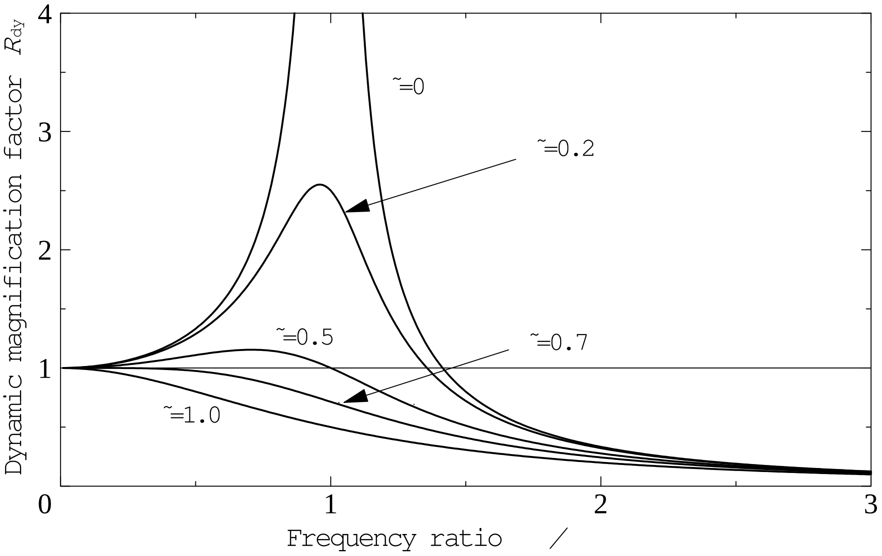


Fig.2.3.2 Dynamic magnification factor R_{dy} with damping ratio ζ and frequency ratio r_{n}

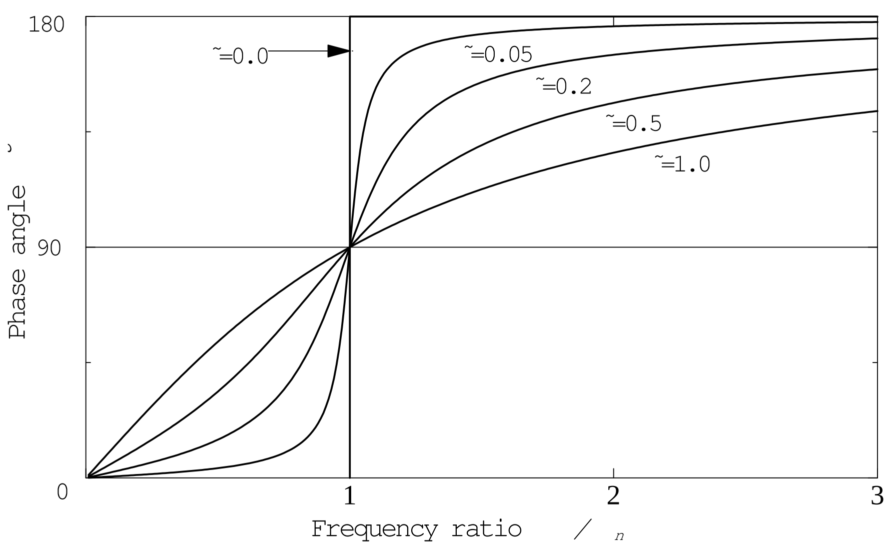


Fig.2.3.3 Phase angle θ with damping ratio ζ and frequency ratio r_{n}

[Example 2.3]

Determine the response of the following system using the applied force prescribed below. The system is initially at rest.

$$m = 3,\ k = 51,\ c = 6$$

$$\begin{aligned} p(t) &= 3\sin 2t \qquad &&\text{for} \quad 0 \le t \le \pi, \\ p(t) &= 0 &&\text{for} \quad t \ge \pi \end{aligned}$$

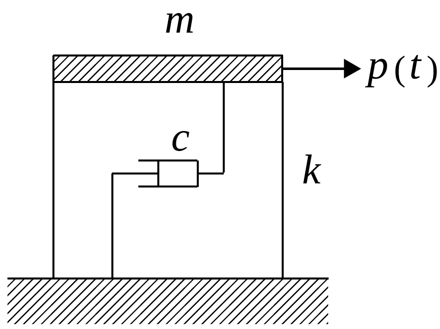


Fig.E2.3

[Solution]

The external load $p(t)$ is given only when $0 \le t \le \pi$. Then the damped free vibration starts at $t = \pi$. The equation of motion is

$$3\ddot{x}(t) + 6\dot{x}(t) + 51x(t) = p(t)$$

The characteristic equation is

$$3s^2 + 6s + 51 = 0$$

Then,

$$s^2 + 2s + 17 = 0$$

Therefore,

$$s = -1 \pm 4i$$

The complementary function is

$$x_{\mathrm{c}}(t) = e^{-t}(A\cos 4t + B\sin 4t)$$

The particular integral can be

$$\begin{aligned} x_{\mathrm{p}}(t) &= C\cos 2t + D\sin 2t \\ \dot{x}_{\mathrm{p}}(t) &= -2C\sin 2t + 2D\cos 2t \\ \ddot{x}_{\mathrm{p}}(t) &= -4C\cos 2t - 4D\sin 2t \end{aligned}$$

Substituting these into the equation of motion, we have,

$$3(-4C\cos 2t - 4D\sin 2t) + 6(-2C\sin 2t + 2D\cos 2t) + 51(C\cos 2t + D\sin 2t) = 3\sin 2t$$

$$(39C + 12D)\cos 2t + (-12C + 39D)\sin 2t = 3\sin 2t$$

Then,

$$\begin{aligned} 39C + 12D &= 0 \\ -12C + 39D &= 3 \end{aligned}$$

Then,

$$C = -\frac{4}{185}, \qquad D = \frac{13}{185}$$

The complete solution is

$$x(t) = e^{-t}(A\cos 4t + B\sin 4t) - \frac{4}{185}\cos 2t + \frac{13}{185}\sin 2t$$

Then,

$$\begin{aligned}\dot{x}(t) = -e^{-t}(A\cos 4t + B\sin 4t) + e^{-t}(-4A\sin 4t + 4B\cos 4t)\\ + \frac{8}{185}\sin 2t + \frac{26}{185}\cos 2t\end{aligned}$$

The system is initially at rest, so that

$$\begin{aligned}x(t=0) = 0 = A - \frac{4}{185}\\ A = \frac{4}{185}\\ \dot{x}(t=0) = 0 = A + 4B + \frac{26}{185}\\ B = \frac{1}{4}(-\frac{26}{185} + \frac{4}{185}) = -\frac{11}{370}\end{aligned}$$

Therefore, for $0 \le t \le \pi$

$$\begin{aligned}x(t) &= \frac{1}{185}\Big\{e^{-t}(4\cos 4t - \frac{11}{2}\sin 4t) - 4\cos 2t + 13\sin 2t\Big\}\\ \dot{x}(t) &= \frac{1}{185}\Big\{-e^{-t}(4\cos 4t - \frac{11}{2}\sin 4t) + e^{-t}(-16\sin 4t - 22\cos 4t) + 8\sin 2t + 26\cos 2t\Big\}\end{aligned}$$

At $t = \pi$

$$\begin{aligned}x(\pi) &= \frac{1}{185}(4e^{-\pi} - 4) = \frac{4}{185}(e^{-\pi} - 1)\\ \dot{x}(\pi) &= \frac{1}{185}(-4e^{-\pi} - 22e^{-\pi} + 26) = -\frac{26}{185}(e^{-\pi} - 1)\end{aligned}$$

These are the boundary conditions for $t \ge \pi$. The complementary function is

$$\begin{aligned}x_c(t) &= e^{-t}(A\cos 4t + B\sin 4t)\\ \dot{x}_c(t) &= -e^{-t}(A\cos 4t + B\sin 4t) + e^{-t}(-4A\sin 4t + 4B\cos 4t)\end{aligned}$$

Substituting the boundary conditions at $t = \pi$

$$\begin{aligned}Ae^{-\pi} &= \frac{4}{185}(e^{-\pi} - 1)\\ A &= \frac{4}{185}(1 - e^{\pi})\\ -Ae^{-\pi} + 4Be^{-\pi} &= -\frac{26}{185}(e^{-\pi} - 1)\\ 4Be^{-\pi} &= -\frac{22}{185}(e^{-\pi} - 1)\\ B &= \frac{11}{370}(e^{\pi} - 1)\end{aligned}$$

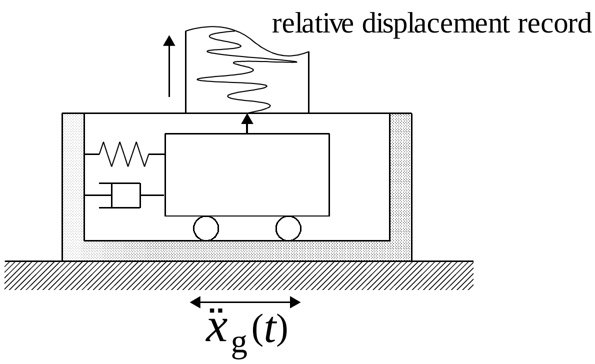


Fig.2.3.4 Scheme of a seismometer

Therefore, for $t \geq \pi$

$$
\begin{aligned}
x(t) &= e^{-t}\{\frac{4}{185}(1-e^{\pi})\cos 4t + \frac{11}{370}(e^{\pi}-1)\sin 4t\} \\
&= \frac{(e^{\pi}-1)}{185}e^{-t}(-4\cos 4t + \frac{11}{2}\sin 4t)
\end{aligned}
$$

(3) Accelerometer and Displacement Meter

A SDOF system mounted in a case is attached to the floor as shown in Fig.2.3.4. The floor is subjected to the ground motion of

$$\ddot{x}_g(t) = p_a \sin \omega t \tag{2.3.23}$$

The equation of motion is

$$m\,\ddot{x}(t) + c\,\dot{x}(t) + k\,x(t) = -mp_a \sin \omega t \tag{2.3.24}$$

The solution for the steady state response of this equation is, as already given by Eq.(2.3.20)

$$
\begin{aligned}
x(t) &= -\frac{mp_a}{k}R_{dy}\sin(\omega t - \theta) \\
&= -\frac{p_a}{\omega_n^2}R_{dy}\sin(\omega t - \theta) \qquad (2.3.25)
\end{aligned}
$$

The dynamic magnification factor R_{dy} is already shown in Fig.2.3.2. From this figure, it can be seen that the dynamic magnification factor is almost unity for $\zeta = 0.7$ and $0 < r_n < 0.6$. Therefore, if the system has a device to record the relative displacement of the system, the record will be proportional to the acceleration amplitude of the ground motions and it can be used as an accelerometer. The frequency of the excitations should be in the range of $0 < r_n < 0.6$. Hence, by increasing the stiffness and/or decreasing the mass, the applicable frequency range will be increased.

If the ground motion is given by

$$x_g(t) = p_d \sin \omega t \tag{2.3.26a}$$

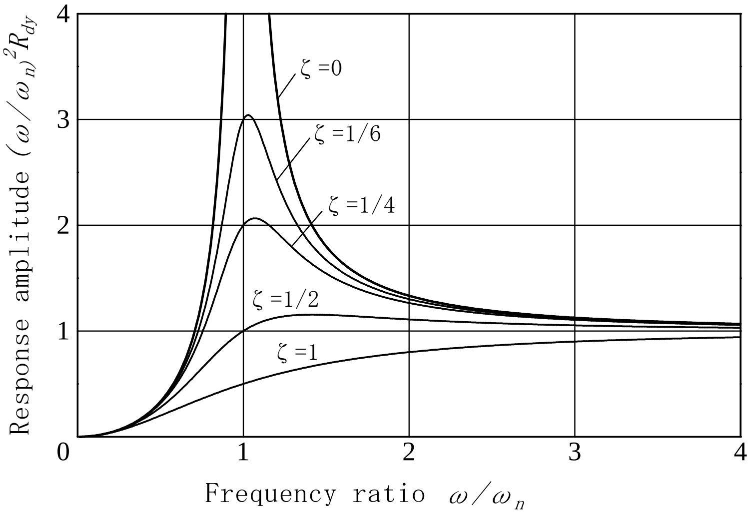


Fig.2.3.5 Response of seismometer to harmonic base displacement

or

$$\ddot{x}_{\mathrm{g}}(t) = -\omega^2 p_{\mathrm{d}} \sin \omega t \tag{2.3.26b}$$

The equation of motion and the solution for steady state response are

$$m\,\ddot{x}(t) + c\,\dot{x}(t) + k\,x(t) = m\omega^2 p_{\mathrm{d}} \sin \omega t \tag{2.3.27}$$

$$\begin{aligned} x(t) &= \frac{m\omega^2 p_{\mathrm{d}}}{k} R_{\mathrm{dy}} \sin(\omega t - \theta) \\ &= r_{\mathrm{n}}^2 R_{\mathrm{dy}} p_{\mathrm{d}} \sin(\omega t - \theta) \end{aligned} \tag{2.3.28}$$

The relations between $r_{\mathrm{n}}^2 R_{\mathrm{dy}}$ and r_{n} are shown in Fig.2.3.5.

It is evident that $r_{\mathrm{n}}^2 R_{\mathrm{dy}}$ is almost constant for $\zeta = 0.5$ and $r_{\mathrm{n}} > 1$. Thus the relative displacement amplitude of the system is proportional to the excitation displacement and the system can be used as a displacement meter. The frequency of the system should be decreased by reducing the stiffness and/or increasing the mass.

(4) Vibration Isolation

A rotating machine produces an oscillatory force $p(t) = p_0 \sin \omega t$ due to unbalance in its rotating parts. If the machine is supported by a spring-damper system, the equation of motion of the machine and its steady state response are (See Fig.2.3.6)

$$m\,\ddot{y}(t) + c\,\dot{y}(t) + k\,y(t) = p_0 \sin \omega t \tag{2.3.29}$$

$$y(t) = \frac{p_0}{k} R_{\mathrm{dy}} \sin(\omega t - \theta) \tag{2.3.30}$$

Thus the force p_s from the spring to the floor is

$$p_s = k\,y(t) = p_0 R_{\mathrm{dy}} \sin(\omega t - \theta) \tag{2.3.31}$$

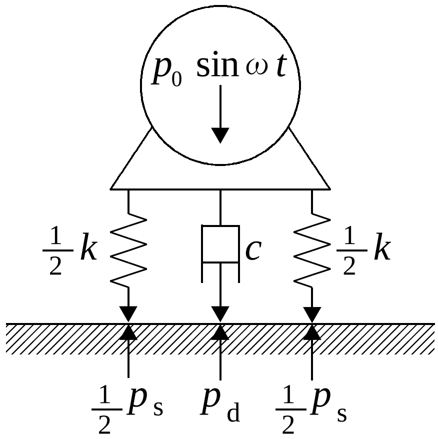


Fig.2.3.6 Vibration isolation system

The relative velocity to the floor is

$$\dot{y}(t) = \frac{p_0}{k} R_{\mathrm{dy}}\, \omega \cos(\omega t - \theta) \tag{2.3.32}$$

Then, the damping force p_{d} is

$$\begin{aligned} p_{\mathrm{d}} &= c\,\dot{y}(t) = \frac{c\,p_0}{k} R_{\mathrm{dy}}\, \omega \cos(\omega t - \theta) \\ &= 2\zeta r_{\mathrm{n}} p_0 R_{\mathrm{dy}}\ \cos(\omega t - \theta) \end{aligned} \tag{2.3.33}$$

Therefore, the force p given to the floor by the machine is

$$p = p_{\mathrm{s}} + p_{\mathrm{d}} = p_0 R_{\mathrm{dy}} \sqrt{1 + (2\zeta r_{\mathrm{n}})^2} \cos(\omega t - \theta - \theta') \tag{2.3.34}$$

The ratio R_{tr} of the maximum force to the applied force amplitude is called transmissibility and it is given by

$$R_{\mathrm{tr}} = \frac{p_{\mathrm{max}}}{p_0} = R_{\mathrm{dy}} \sqrt{1 + (2\zeta r_{\mathrm{n}})^2}$$

Let us consider the case of $R_{\mathrm{tr}} = 1$. In this case,

$$\begin{aligned} 1 + (2\zeta r_{\mathrm{n}})^2 &= (1 - r_{\mathrm{n}}^2)^2 + (2\zeta r_{\mathrm{n}})^2 \\ r_{\mathrm{n}}^2 (r_{\mathrm{n}}^2 - 2) &= 0 \end{aligned}$$

Therefore, regardless of ζ, when $r_{\mathrm{n}} = 0$ or $\sqrt{2}$, $R_{\mathrm{tr}} = 1$. A plot of R_{tr} is shown in Fig.2.3.7. It is similar to Fig.2.3.2, but all the curves pass through the same point at $r_{\mathrm{n}} = \sqrt{2}$. It can be seen that the given force is reduced for $r_{\mathrm{n}} > \sqrt{2}$ and that the damping reduces the effectiveness of the vibration isolation for $r_{\mathrm{n}} > \sqrt{2}$. If the machine gradually increases the frequency from rest, the machine will be in resonance before it reaches its steady state response. This resonance may cause unfavorable vibration of the floor as well as of the machine itself. Therefore the optimum point should be selected in order to design the vibration isolation system.

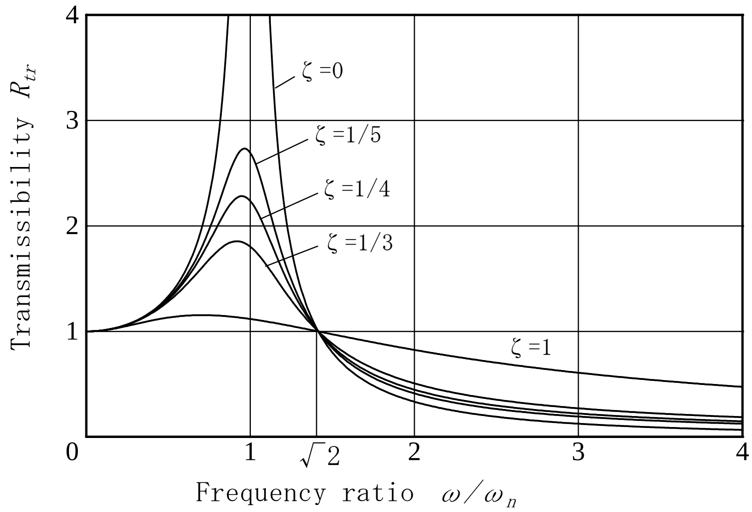


Fig.2.3.7 Vibration transmissibility ratio

2.4 Response to Arbitrary Loading

(1) Linear Acceleration Method

If the force applied is arbitrary, we usually use step-by-step methods to calculate the response of a system. One of the advantages of the step-by-step method is that it is applicable not only to elastic systems but also to inelastic systems. Here, we are going to discuss the linear acceleration method, which is one of the step-by-step methods.

In order to study this method, let us start with the equation of motion of a SDOF system that we have already learned, i.e.

$$-m\,\ddot{x}_T(t) - c\,\dot{x}(t) - k\,x(t) = 0 \tag{2.4.1}$$

where $\ddot{x}_T(t)$ is the absolute acceleration and it is

$$\ddot{x}_T(t) = \ddot{x}(t) + \ddot{x}_{\mathrm{g}}(t) \tag{2.4.2}$$

We should note that only the inertia force is expressed by the term of absolute quantity in Eq.(2.4.1). Substituting Eq.(2.4.2) into Eq.(2.4.1) and after some rearrangement, we have

$$\ddot{x}(t) + \frac{c}{m}\dot{x}(t) + \frac{k}{m}x(t) = -\ddot{x}_{\mathrm{g}}(t) \tag{2.4.3}$$

This can be written in the form

$$\ddot{x}(t) + 2\zeta\omega_{\mathrm{n}}\,\dot{x}(t) + \omega_{\mathrm{n}}^2 x(t) = -\ddot{x}_{\mathrm{g}}(t) \tag{2.4.4}$$

where $\zeta = \dfrac{c}{2\sqrt{mk}}$ and $\omega_{\mathrm{n}}^2 = \dfrac{k}{m}$, from their definitions.

Eq.(2.4.4) must be satisfied at any time, i.e. $t = t$. At $t = t + \Delta t$, Eq.(2.4.4) becomes

$$\ddot{x}(t+\Delta t) + 2\zeta\omega_{\mathrm{n}}\dot{x}(t+\Delta t) + \omega_{\mathrm{n}}^2 x(t+\Delta t) = -\ddot{x}_{\mathrm{g}}(t+\Delta t) \tag{2.4.5}$$

Then, let us remember Taylor series expansion, i.e.

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + f'''(x)\frac{h^3}{3!} + \cdots$$

Therefore, we can write

$$x(t+\Delta t) = x(t) + \dot{x}(t)\Delta t + \ddot{x}(t)\frac{\Delta t^2}{2} + \dddot{x}(t)\frac{\Delta t^3}{6} + \cdots \tag{2.4.6a}$$

$$\dot{x}(t+\Delta t) = \dot{x}(t) + \ddot{x}(t)\Delta t + \dddot{x}(t)\frac{\Delta t^2}{2} + \cdots \tag{2.4.7a}$$

If acceleration changes linearly with time during Δt, we have

$$\dddot{x}(t) = \frac{\ddot{x}(t+\Delta t) - \ddot{x}(t)}{\Delta t} \tag{2.4.8}$$

Substituting this into Eqs. (2.4.6) and (2.4.7),

$$x(t+\Delta t) = x(t) + \dot{x}(t)\Delta t + \ddot{x}(t)\frac{\Delta t^2}{3} + \ddot{x}(t+\Delta t)\frac{\Delta t^2}{6} \tag{2.4.6b}$$

$$\dot{x}(t+\Delta t) = \dot{x}(t) + \ddot{x}(t)\frac{\Delta t}{2} + \ddot{x}(t+\Delta t)\frac{\Delta t}{2} \tag{2.4.7b}$$

Substituting these equations into Eq.(2.4.5), we have

$$\begin{aligned}
&\ddot{x}(t+\Delta t) + 2\zeta\omega_{\mathrm{n}}\Big\{\dot{x}(t) + \ddot{x}(t)\frac{\Delta t}{2} + \ddot{x}(t+\Delta t)\frac{\Delta t}{2}\Big\} \\
&\qquad + \omega_{\mathrm{n}}^2\Big\{x(t) + \dot{x}(t)\Delta t + \ddot{x}(t)\frac{\Delta t^2}{3} + \ddot{x}(t+\Delta t)\frac{\Delta t^2}{6}\Big\} = -\ddot{x}_{\mathrm{g}}(t+\Delta t) \\
&\ddot{x}(t+\Delta t)\Big\{1 + \zeta\omega_{\mathrm{n}}\Delta t + \frac{\omega_{\mathrm{n}}^2\Delta t^2}{6}\Big\} + x(t)\omega_{\mathrm{n}}^2 \\
&\qquad + \dot{x}(t)\{2\zeta\omega_{\mathrm{n}} + \omega_{\mathrm{n}}^2\Delta t\} + \ddot{x}(t)\Big\{\zeta\omega_{\mathrm{n}}\Delta t + \frac{\omega_{\mathrm{n}}^2}{3}\Delta t^2\Big\} = -\ddot{x}_{\mathrm{g}}(t+\Delta t)
\end{aligned}$$

Therefore,

$$\ddot{x}(t+\Delta t) = -\frac{1}{A}\{\ddot{x}_{\mathrm{g}}(t+\Delta t) + x(t)B + \dot{x}(t)C + \ddot{x}(t)D\} \tag{2.4.9}$$

where,

$$A = \{1 + \zeta\omega_{\mathrm{n}}\Delta t + \frac{\omega_{\mathrm{n}}^2}{6}\Delta t^2\}, \quad B = \omega_{\mathrm{n}}^2, \quad C = \{2\zeta\omega_{\mathrm{n}} + \omega_{\mathrm{n}}^2\Delta t\}$$

and

$$D = \{\zeta\omega_{\mathrm{n}}\Delta t + \frac{\omega_{\mathrm{n}}^2}{3}\Delta t^2\}$$

and these are independent of time. Therefore we do not have to calculate these at each step, as long as the system remains in the elastic range. In the case the system is initially at rest, $x(0) = \dot{x}(0) = 0$, then at $t = \Delta t$

$$\ddot{x}(\Delta t) = -\frac{1}{A}\ddot{x}_{\mathrm{g}}(\Delta t)$$

Substituting this into Eqs. (2.4.6b) and (2.4.7b), we know $x(\Delta t)$ and $\dot{x}(\Delta t)$. Then we can calculate $\ddot{x}(2\Delta t)$ and then $\dot{x}(2\Delta t)$ and $x(2\Delta t)$, and so on. Therefore, we will know the whole response of the system. In the case of earthquake acceleration, Δt is usually 0.01 or 0.02 (s).

[**Example 2.4**]

In case you compute the response of a SDOF system to earthquake motions, you will probably need the aid of a digital computer because there are tens of thousands of calculations. Here we will show you an example of a computer program in FORTRAN language for response computation of a SDOF system.

A Fortran Program for computing the response to equi-spaced earthquake motion data for a SDOF system

```
C
C     Program for computing the response to earthquake
C     motion
C
C     Description of the parameters
C         A(I)   -  Accelerogram in gals
C         DAMP   -  Fraction of critical damping
C         PER    -  Natural period in second of the SDOF system
C         DT     -  Time interval in second of accelerogram
C         NN     -  Number of data in accelerogram
C
      REAL*4 A(3000),RA(3000),RV(3000),RD(3000)
      READ(5,500) DAMP,PER
  500 FORMAT(2F10.5)
      READ(5,501) DT,NN
  501 FORMAT(F10.0,I5)
      READ(5,502) (A(M),M=1,NN)
  502 FORMAT(10F8.0)
C
C     Computing the response
C
      CALL RESP(PER,DAMP,NN,DT,A,RA,RV,RD)
C     Computing the maximum of the responses
C
      SA=0.
      SV=0.
      SD=0.
      DO 100 M=1,NN
      IF (SA.GT.ABS(RA)) GO TO 100
      SA=ABS(RA)
      TA=DT*(M-1)
```

```
  100 IF (SV.GT.ABS(RV)) GO TO 200
      SV=ABS(RV)
      TV=DT*(M-1)
  200 IF (SD.GT.ABS(RD)) GO TO 300
      SD=ABS(RD)
      TD=DT*(M-1)
  300 CONTINUE
      WRITE(6,600) SA,SV,SD
  600 FORMAT(1H1/1H0/1H ,10X,'MAX. ACC. RESPONSE(GAL) =',
     1         F10.3/1H ,10X,'MAX. VEL. RESPONSE(KINE) =',
     2         F10.4/1H ,10X,'MAX. DIS. RESPONSE(CM) =',
     3         F10.5)
      STOP
      END
C
C
C     Subroutine program to compute the response time
C     history of the given damping ratio and given natural
C     period for given acceleration time history
C     INPUT
C       PER  - NATURAL PERIOD OF THE SDOF SYSTEM
C       DAMP - FRACTION OF CRITICAL DAMPING
C       NN   - NUMBER OF DATA
C       DT   - TIME INTERVAL OF DATA
C       A    - ACCELEROGRAM FOR WHICH RESPONSE IS COMPUTED
C     OUTPUT
C       RA   - ABSOLUTE ACC. RESPONSE (ARRAY)
C       RV   - RELATIVE VEL. RESPONSE (ARRAY)
C       RD   - RELATIVE DIS. RESPONSE (ARRAY)
C
      SUBROUTINE RESP(PER,DAMP,NN,DT,A,RA,RV,RD)
C
      REAL*4 A(1),RA(1),RV(1),RD(1)
      DATA PI2/6.2831853/
      W=6.283185/PER
      R=1.0+DAMP*W*DT+(W*DT)**2/6.0
      DDX=(2.0*DAMP*W*DT-1.0)*A(1)
      DX=-A(1)*DT
      X=0.0
      RA(1)=DDX+A(1)
      RV(1)=DX
      RD(1)=0.0
C
C     RESPONSE COMPUTATIONS
C
```

```
      DO 110 M=2,NN
      E=DX+DDX*DT/2.0
      F=X+DX*DT+DDX*DT**2/3.0
      DDX=-(A(M)+2.0*DAMP*W*E+W**2*F)/R
      DX=E+DDX*DT/2.0
      X=F+DDX*DT**2/6.0
      RA(M)=DDX+A(M)
      RV(M)=DXVELMAX
      RD(M)=X
  110 CONTINUE
      RETURN
      END
```

(2) Duhamel Integral - Convolution Integral

If a force p is applied to a system for a short duration of Δt, the applied impulse is $p\Delta t$, and this is equal to the increment of momentum of the system which is given by $m\,\Delta\dot{x}$. Thus,

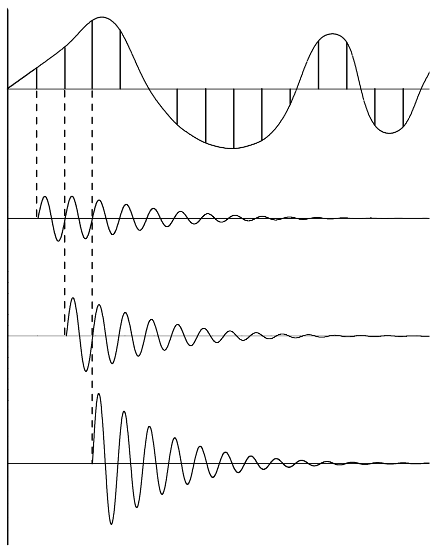
$$p\Delta t = m\,\Delta\dot{x} \tag{2.4.10}$$

The free vibration of the system, which is initially at rest, due to the impulse $p\Delta t$ or initial velocity $p\Delta t/m$ is [See Eq.(2.2.22b)]

$$x(t) = e^{-\zeta\omega_{\mathrm{n}}t}\frac{p\Delta t}{m\omega_{\mathrm{d}}}\sin\omega_{\mathrm{d}}t \tag{2.4.11}$$

Thus the response due to the arbitrary load $p(t)$ is (See Fig.2.4.1)

$$x(t) = \frac{1}{m\omega_{\mathrm{d}}}\int_0^t p(\tau)e^{-\zeta\omega_{\mathrm{n}}(t-\tau)}\sin\omega_{\mathrm{d}}(t-\tau)d\tau \tag{2.4.12}$$



$p(t)$

$$x(t) = e^{-\zeta\omega_{\mathrm{n}}(t-\Delta t)}\frac{p(\Delta t)\Delta t}{m\omega_{\mathrm{d}}}\sin\omega_{\mathrm{d}}(t-\Delta t)$$

$$x(t) = e^{-\zeta\omega_{\mathrm{n}}(t-2\Delta t)}\frac{p(2\Delta t)\Delta t}{m\omega_{\mathrm{d}}}\sin\omega_{\mathrm{d}}(t-2\Delta t)$$

$$x(t) = e^{-\zeta\omega_{\mathrm{n}}(t-3\Delta t)}\frac{p(3\Delta t)\Delta t}{m\omega_{\mathrm{d}}}\sin\omega_{\mathrm{d}}(t-3\Delta t)$$

Fig.2.4.1 Convolution integral

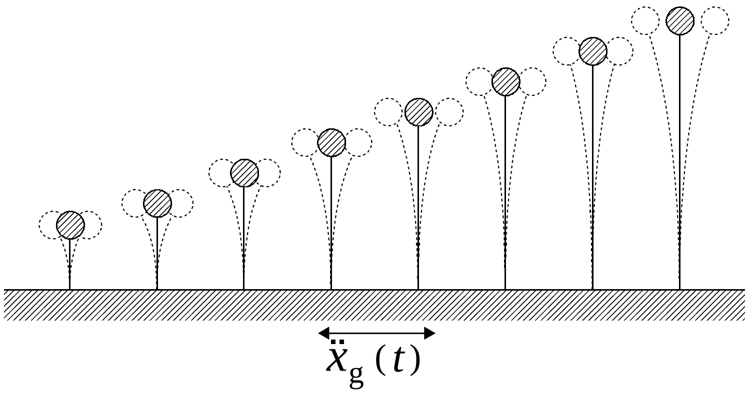


Fig.2.4.2 Scheme of response spectrum

This is known as the Duhamel integral that is also expressed in the form

$$x(t) = \int_0^t p(\tau)h(t-\tau)d\tau \tag{2.4.13}$$

This form is called the convolution integral and the new symbol is defined by

$$h(t-\tau) = \frac{1}{m\omega_\mathrm{d}} e^{-\zeta\omega_\mathrm{n}(t-\tau)} \sin\omega_\mathrm{d}(t-\tau) \tag{2.4.14}$$

This is called unit-impulse response.

If the force is caused by an external acceleration $\ddot{x}_\mathrm{g}(t)$, Eq.(2.4.12) becomes

$$x(t) = -\frac{1}{\omega_\mathrm{d}} \int_0^t \ddot{x}_\mathrm{g}(\tau) e^{-\zeta\omega_\mathrm{n}(t-\tau)} \sin\omega_\mathrm{d}(t-\tau)d\tau \tag{2.4.15}$$

(3) Response Spectrum

The intensity of the ground motion gives engineers valuable information related to the extent of damage of structures. The most important properties of earthquake ground motion records are amplitude, frequency content and duration of the motion. By considering these three important properties, a convenient measure of a ground motion can be obtained by evaluating the response of SDOF systems (see Fig.2.4.2). This measure is called the response spectrum, which can be obtained as follows.

Structural response to earthquake excitations varies according to the dynamic characteristics of the system. The dynamic characteristics are represented simply using only damping and natural period for the simplest structure, i.e. the SDOF system.

As was described in the previous section, the response of the structure changes with time. In structural design, in most cases the maximum response is more important than its time variation. The maximum response to an earthquake input motion is a function of the damping ratio ζ, and the natural period T. Let the maximum responses for relative displacement, relative velocity and absolute acceleration be represented by $S_\mathrm{d}(\zeta, T)$,

$S_{\mathrm{v}}(\zeta, T)$ and $S_{\mathrm{a}}(\zeta, T)$, respectively. They can be calculated by using the step-by-step method or by using Duhamel's integration method as follows.

$$S_{\mathrm{d}}(\zeta, T) = \frac{1}{\omega_{\mathrm{d}}}\Big|\int_0^t \ddot{x}_{\mathrm{g}}(\tau)e^{-\zeta\omega_{\mathrm{n}}(t-\tau)}\sin\omega_{\mathrm{d}}(t-\tau)d\tau\Big|_{\max} \tag{2.4.16a}$$

$$S_{\mathrm{v}}(\zeta, T) = \Big|\int_0^t \ddot{x}_{\mathrm{g}}(\tau)e^{-\zeta\omega_{\mathrm{n}}(t-\tau)}[\cos\omega_{\mathrm{d}}(t-\tau) - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_{\mathrm{d}}(t-\tau)]d\tau\Big|_{\max} \tag{2.4.16b}$$

$$S_{\mathrm{a}}(\zeta, T) = \omega_{\mathrm{d}}\Big|\int_0^t \ddot{x}_{\mathrm{g}}(\tau)e^{-\zeta\omega_{\mathrm{n}}(t-\tau)}[(1-\frac{\zeta^2}{1-\zeta^2})\sin\omega_{\mathrm{d}}(t-\tau) + \frac{2\zeta}{\sqrt{1-\zeta^2}}\cos\omega_{\mathrm{d}}(t-\tau)]d\tau\Big|_{\max} \tag{2.4.16c}$$

The figure that presents the relationship between maximum response and period is called the response spectrum. Therefore, $S_{\mathrm{d}}(\zeta, T), S_{\mathrm{v}}(\zeta, T)$ and $S_{\mathrm{a}}(\zeta, T)$ are called the relative displacement response spectrum, relative velocity spectrum and absolute acceleration spectrum, respectively.

In the case of structural design, several strong motion records are commonly used. The record at El Centro during the Imperial Valley earthquake, California in 1940 is the most popular one in the world. The response spectra for this record are shown in Fig.2.4.3. By analyzing many time history records from earthquakes, it has been found that the response velocity of structures becomes almost constant for longer periods. Therefore a response spectrum has the characteristics shown in Fig.2.4.4.

[Pseudo response spectra]

Since the damping ratio for ordinary structures is far less than 1, it will be found that

$$\zeta^2 \approx 0 \ , \quad \sqrt{1-\zeta^2} \approx 1 \quad \text{and} \quad \omega_{\mathrm{d}} \approx \omega_{\mathrm{n}} = \frac{2\pi}{T}$$

Then, Eqs.(2.4.16a-c) can be rewritten approximately as follows:

$$S_{\mathrm{d}}(\zeta, T) \approx \frac{1}{\omega_{\mathrm{n}}}\Big|\int_0^t \ddot{x}_{\mathrm{g}}(\tau)e^{-\zeta\omega_{\mathrm{n}}(t-\tau)}\sin\omega_{\mathrm{n}}(t-\tau)d\tau\Big|_{\max} \tag{2.4.17a}$$

$$S_{\mathrm{v}}(\zeta, T) \approx \Big|\int_0^t \ddot{x}_{\mathrm{g}}(\tau)e^{-\zeta\omega_{\mathrm{n}}(t-\tau)}\cos\omega_{\mathrm{n}}(t-\tau)d\tau\Big|_{\max} \tag{2.4.17b}$$

$$S_{\mathrm{a}}(\zeta, T) \approx \omega_{\mathrm{n}}\Big|\int_0^t \ddot{x}_{\mathrm{g}}(\tau)e^{-\zeta\omega_{\mathrm{n}}(t-\tau)}\sin\omega_{\mathrm{n}}(t-\tau)d\tau\Big|_{\max} \tag{2.4.17c}$$

Let us introduce a new symbol as

$$S_{\mathrm{pv}}(\zeta, T) = \Big|\int_0^t \ddot{x}_{\mathrm{g}}(\tau)e^{-\zeta\omega_{\mathrm{n}}(t-\tau)}\sin\omega_{\mathrm{n}}(t-\tau)d\tau\Big|_{\max} \tag{2.4.18b}$$

It has been proved that $S_{\mathrm{pv}}(\zeta, T)$ differs very little from $S_{\mathrm{v}}(\zeta, T)$. Introducing two more new symbols $S_{\mathrm{pd}}(\zeta, T)$, and $S_{\mathrm{pa}}(\zeta, T)$ and using Eq.(2.4.18b), Eqs.(2.4.17a) and (2.4.17c) becomes as follows.

$$S_{\mathrm{pd}}(\zeta, T) = \frac{1}{\omega_{\mathrm{n}}}S_{\mathrm{pv}}(\zeta, T) \approx S_{\mathrm{d}}(\zeta, T) \tag{2.4.18a}$$

$$S_{\mathrm{pa}}(\zeta, T) = \omega_{\mathrm{n}}S_{\mathrm{pv}}(\zeta, T) \approx S_{\mathrm{a}}(\zeta, T) \tag{2.4.18c}$$

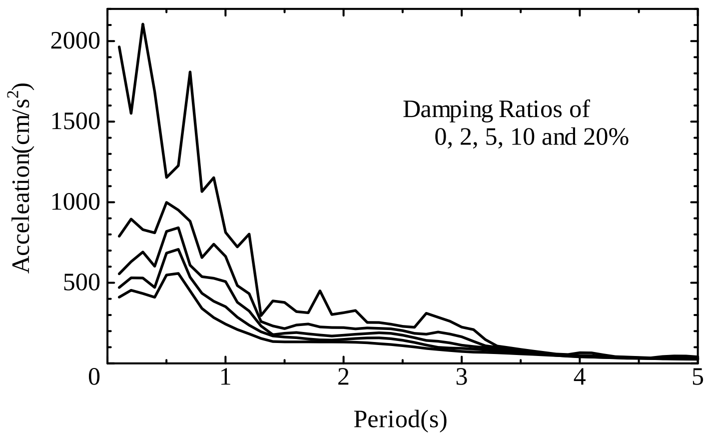


Fig.2.4.3(a) Acceleration response spectrum
(El Centro 1940 NS)

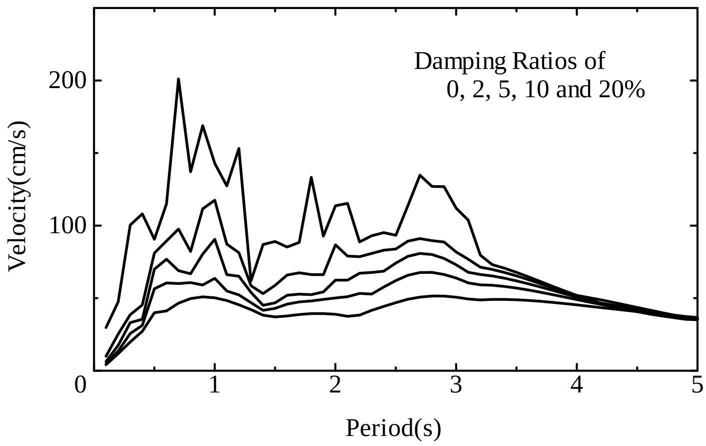


Fig.2.4.3(b) Velocity response spectrum
(El Centro 1940 NS)

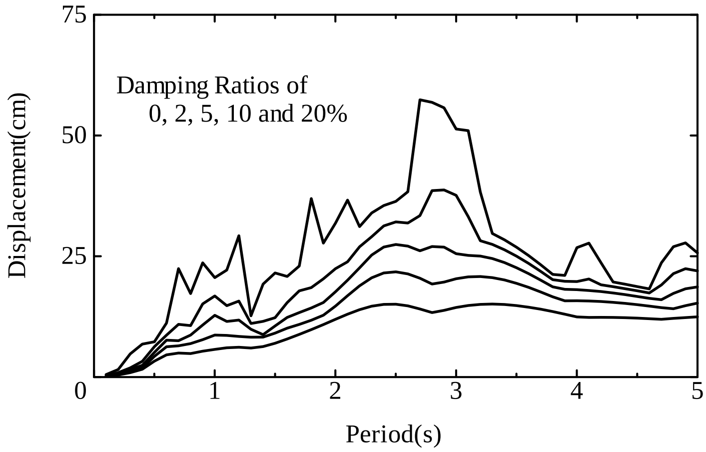


Fig.2.4.3(c) Displacement response spectrum
(El Centro 1940 NS)

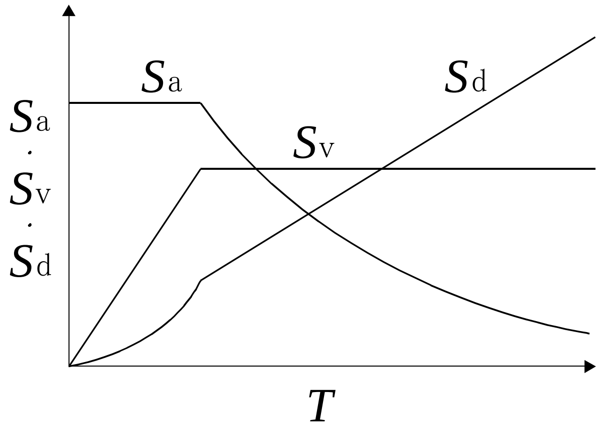


Fig.2.4.4 Typical characteristics of response spectra

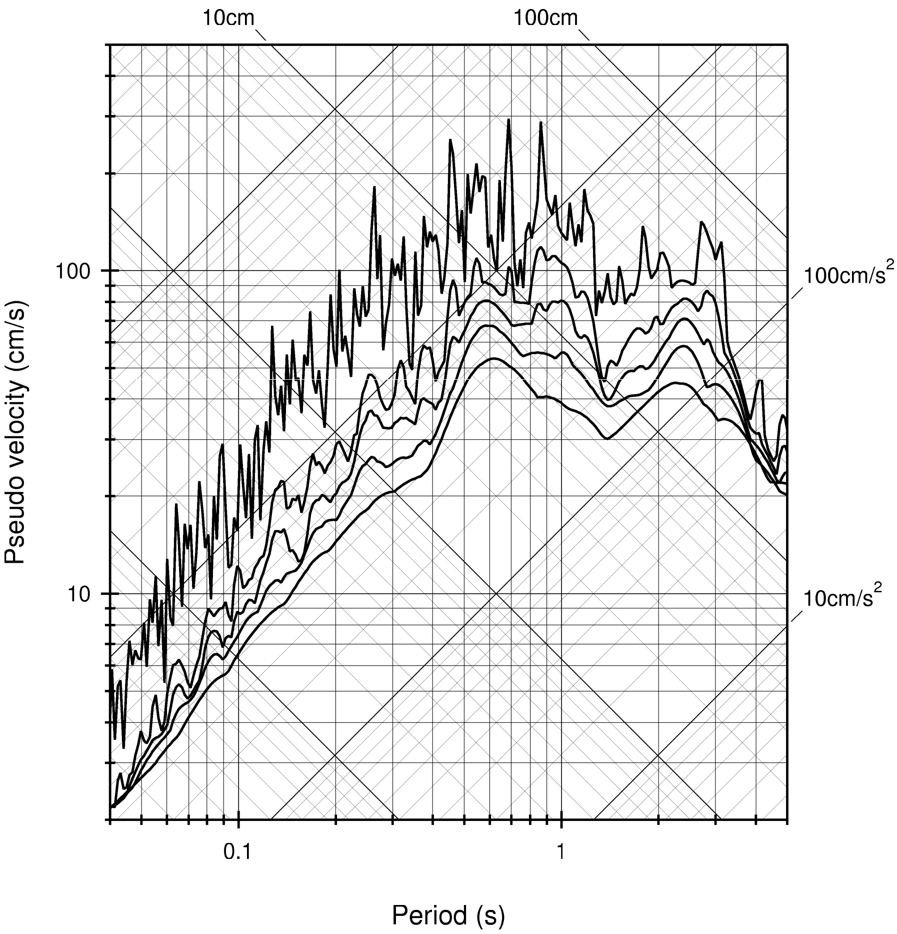


Fig.2.4.5 Tripartite response spectrum
(El Centro 1940 NS, ζ =0, 0.02, 0.05, 0.1, 0.2)

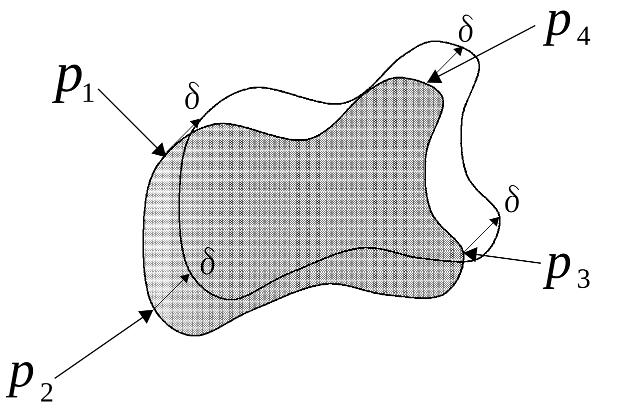


Fig.2.5.1 Virtual displacement of a body

The values $S_{\text{pd}}(\zeta, T)$, $S_{\text{pv}}(\zeta, T)$, and $S_{\text{pa}}(\zeta, T)$ are called the pseudo (relative) displacement response, pseudo (relative) velocity response and pseudo (absolute) acceleration response, respectively.

Due to these simple relationships, it is possible to present the three types of responses in a single plot. A four-way log plot as shown in Fig.2.4.5 allows the three types of spectra on a single graph that is called a tripartite response spectrum.

If the response spectrum is given, the maximum response of any SDOF system can be determined. Furthermore, it should be noted that the response of a multi-degree of freedom (MDOF) system can also be approximately evaluated by using the response spectrum. The method, e.g. modal analysis or the square root of sum of squares (SRSS) method, will be explained in Chapter 3.

2.5 Principle of Virtual Work - Generalized SDOF Systems

If the structural system is complicated, the direct equilibration of forces may be difficult. In such cases, the principle of virtual work can be used to formulate the equation of motion as a substitute for the equilibrium relationships.

The principle of virtual work can be expressed as follows. If a system which is in equilibrium under the action of a set of forces is subjected to a virtual displacement, the total work done by the forces will be zero.

(1) For example, suppose there is a body in equilibrium with applied forces, p_1, p_2, p_3 and p_4 (See Fig.2.5.1). Then the work caused by the virtual displacement δ will be zero. Therefore we will have the following equation.

$$p_1 \cdot \delta + p_2 \cdot \delta + p_3 \cdot \delta + p_4 \cdot \delta = 0 \tag{2.5.1}$$

(2) If we apply this principle to a common SDOF system, the following equation can be given. (See Fig.2.5.2)

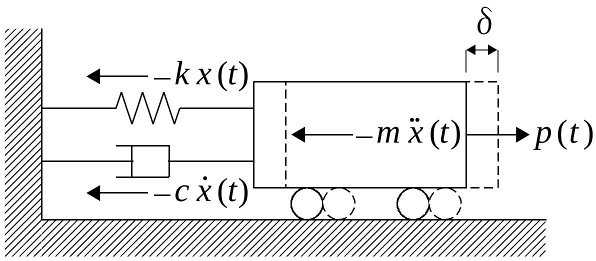


Fig.2.5.2 Model of SDOF systems

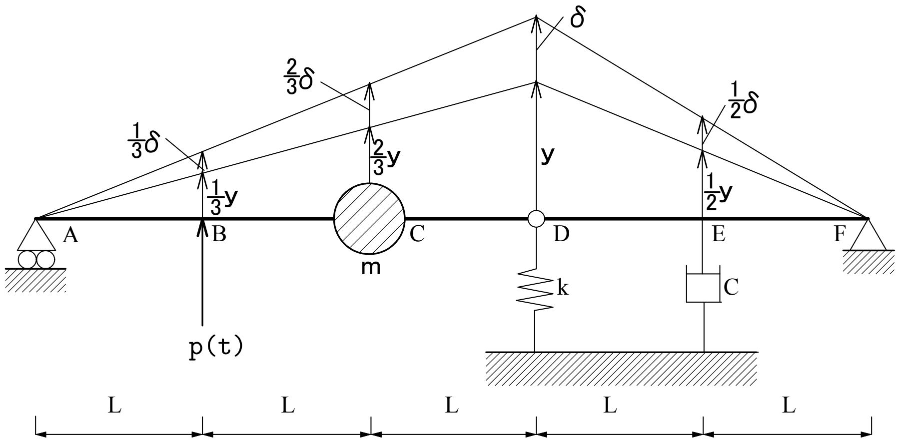


Fig.2.5.3 Virtual displacements of the system

$$-m\,\ddot{x}(t)\cdot\delta - c\,\dot{x}(t)\cdot\delta - k\,x(t)\cdot\delta + p(t)\cdot\delta = 0 \tag{2.5.2}$$

Since δ can be arbitrary, we then have the following equation of motion, which is exactly the same as the equation derived by direct equilibration using d'Alembert's principle.

$$-m\,\ddot{x}(t) - c\,\dot{x}(t) - k\,x(t) + p(t) = 0 \tag{2.5.3}$$

(3) Let us consider the relatively complicated system shown in Fig.2.5.3.

The forces acting on the system at B, C, D and E are $p(t)$, $-m\frac{2}{3}\ddot{y}(t)$, $-k\,y(t)$ and $-c\frac{1}{2}\dot{y}(t)$, respectively, where the vertical displacement at D is defined as $y(t)$. If the virtual displacement δ is given at D, the virtual work caused by this displacement should be zero. Then the following equation can be given.

$$p(t)\cdot\frac{1}{3}\delta - m\frac{2}{3}\ddot{y}(t)\cdot\frac{2}{3}\delta - k\,y(t)\cdot\delta - c\frac{1}{2}\dot{y}(t)\cdot\frac{1}{2}\delta = 0 \tag{2.5.4}$$

Finally, we can have the following equation of motion

$$\left(\frac{2}{3}\right)^2 m\,\ddot{y}(t) + \left(\frac{1}{2}\right)^2 c\,\dot{y}(t) + k\,y(t) = \frac{1}{3}p(t) \tag{2.5.5}$$

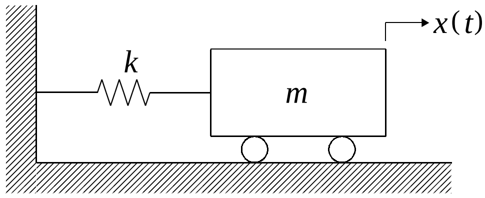


Fig.2.6.1 Free vibration of an undamped spring-mass system

If we denote,

$$m^* = \frac{4}{9}m, \qquad c^* = \frac{1}{4}c, \qquad k^* = k, \qquad \text{and} \qquad p^* = \frac{1}{3}p$$

Eq.(2.5.5) can be written in the form

$$m^*\ddot{y}(t) + c^*\dot{y}(t) + k^*y(t) = p^*(t) \tag{2.5.6}$$

This has the same form as the equation for a simple SDOF system. m^*, c^*, k^* and $p^*(t)$ are called the generalized mass, generalized damping coefficient, generalized stiffness and generalized load, respectively.

It should be noted that the sign of each term of Eqs.(2.5.2) and (2.5.4) is positive when the virtual displacement and the force have the same direction, and is negative when the directions of the virtual displacement and the force are opposite.

2.6 Rayleigh's Method - Vibration of Continuous Members

(1) Basis for the Method

The vibration frequency or period of a SDOF system has a controlling influence on its dynamic behavior. For this reason it is useful to develop a simple method of evaluating the vibration frequency for SDOF systems. Probably the most useful procedure is Rayleigh's method. The basic concept in the Rayleigh's method is the principle of conservation of energy; the energy in a freely vibrating system must remain constant if no damping forces act to absorb it. Let us consider the free vibration motion of an undamped spring-mass system. (See Fig.2.6.1)

The displacement can be expressed by

$$x(t) = x_0 \sin \omega_{\mathrm{n}} t \tag{2.6.1}$$

and the velocity by

$$\dot{x}(t) = x_0\, \omega_{\mathrm{n}} \cos \omega_{\mathrm{n}} t \tag{2.6.2}$$

The potential energy of this system is represented by the strain energy of the spring:

$$E_{\mathrm{P}}(t) = \frac{1}{2}k\, x(t)^2 = \frac{1}{2}k\, x_0^2 \sin^2 \omega_{\mathrm{n}} t \tag{2.6.3a}$$

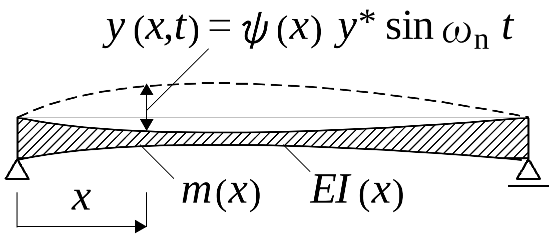


Fig.2.6.2 Vibration of a nonuniform simple beam

while the kinetic energy of the mass is

$$E_{\mathrm{K}}(t) = \frac{1}{2} m\,\dot{x}(t)^2 = \frac{1}{2} m\,x_0^2\,\omega_{\mathrm{n}}^2 \cos^2 \omega_{\mathrm{n}} t \tag{2.6.4a}$$

At the time when $t = T/4 = \pi/(2\omega_{\mathrm{n}})$, the kinetic energy is zero and the potential energy reaches its maximum value:

$$E_{\mathrm{Pmax}} = \frac{1}{2} k\,x_0^2 \tag{2.6.3b}$$

Similarly, at the time when $t = T/2 = \pi/\omega_{\mathrm{n}}$, the potential energy vanishes and the kinetic energy becomes maximum.

$$E_{\mathrm{Kmax}} = \frac{1}{2} m\,x_0^2\,\omega_{\mathrm{n}}^2 \tag{2.6.4b}$$

Hence, if the total energy in the vibrating system remains constant (as it must in undamped free vibration), it is then apparent that the maximum potential energy must be equal to the maximum kinetic energy, $E_{\mathrm{Pmax}} = E_{\mathrm{Kmax}}$. We then have

$$\omega_{\mathrm{n}}^2 = \frac{k}{m} \tag{2.6.5}$$

This is, of course, the same frequency expression which has been given earlier; in this case it has been derived by the Rayleigh concept of equating expressions from the maximum strain energy and the maximum kinetic energy.

(2) Approximate Analysis of General Systems

There is no advantage to be gained from the application of Rayleigh's method to vibration analysis of a spring-mass system as described above; its principal use is for the approximate frequency analysis of a system having many degrees of freedom.

Consider, for example, the non-uniform simple beam shown in Fig.2.6.2. This beam actually has an infinite number of degrees of freedom. In order to apply the Rayleigh procedure, it is necessary to make an assumption about the shape that the beam will take in its fundamental mode of vibration. Noting the harmonic vibration of the generalized coordinate in free vibration, the displacement of the beam can be expressed by

$$y(x,t) = \psi(x)\,y^* \sin \omega_{\mathrm{n}} t \tag{2.6.6}$$

where $\psi(x)$ is the shape function and y^* is the generalized coordinate. This equation expresses the assumption that shape of the vibrating beam does not change with time;

only the amplitude of motion varies, and it varies harmonically in free-vibration. The assumption of the shape function $\psi(x)$ effectively reduces the beam to a SDOF system. The frequency of vibration can be found as follows.

The strain energy of this flexural system is given by

$$E_{\mathrm{P}}(t) = \frac{1}{2}\int EI(x)\left\{\frac{\partial^2 y(x,t)}{\partial x^2}\right\}^2 dx \tag{2.6.7a}$$

Substituting Eq.(2.6.6) and letting the displacement amplitude take its maximum value leads to

$$E_{\mathrm{Pmax}} = \frac{1}{2}(y^*)^2 \int EI(x)\{\psi''(x)\}^2 dx \tag{2.6.7b}$$

The kinetic energy of the nonuniformly distributed mass is

$$E_{\mathrm{K}}(t) = \frac{1}{2}\int m(x)\{\dot{y}(x,t)\}^2 dx \tag{2.6.8a}$$

Differentiating Eq.(2.6.6) and substituting this to Eq.(2.6.8a), the maximum kinetic energy can be given by

$$E_{\mathrm{Kmax}} = \frac{1}{2}(y^*)^2\omega_{\mathrm{n}}^2 \int m(x)\{\psi(x)\}^2 dx \tag{2.6.8b}$$

Equating the maximum potential energy to the maximum kinetic energy, the frequency is found to be

$$\omega_{\mathrm{n}}^2 = \frac{\int EI(x)\{\psi''(x)\}^2 dx}{\int m(x)\{\psi(x)\}^2 dx} \tag{2.6.9}$$

It may be noted that the numerator is merely the generalized stiffness k^* for this assumed displacement shape, while the denominator is its generalized mass m^*.

$$k^* = \int EI(x)\{\psi''(x)\}^2 dx \tag{2.6.10}$$

$$m^* = \int m(x)\{\psi(x)\}^2 dx \tag{2.6.11}$$

Thus Rayleigh's method can be directly applied to any SDOF system of the generalized form and the frequency can be given by

$$\omega_{\mathrm{n}}^2 = \frac{k^*}{m^*} \tag{2.6.12}$$

[Example 2.5]

Derive the natural frequency of a simply supported uniform beam, using Rayleigh's method.

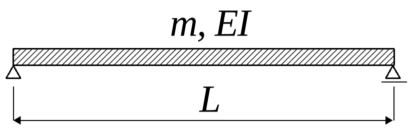


Fig.E2.5

[Solution]

At first, let us assume the shape function (1) as parabolic.

Shape function (1)

$$y(x) = x^2 - Lx \qquad y'(x) = 2x - L \qquad y''(x) = 2$$

Then,

$$\begin{aligned}
k^* &= \int_0^L EI(2)^2 dx = 4EI\Big[x\Big]_0^L = 4EIL \\
m^* &= \int_0^L m\,(x^2 - Lx)^2 dx = m\int_0^L (x^4 - 2Lx^3 + L^2x^2)dx \\
&= m\Big[\frac{x^5}{5} - 2L\frac{x^4}{4} + L^2\frac{x^3}{3}\Big]_0^L = mL^5\Big(\frac{1}{5} - \frac{1}{2} + \frac{1}{3}\Big) = \frac{mL^5}{30} \\
\omega_{\mathrm{n}}^2 &= \frac{k^*}{m^*} = 4EIL\frac{30}{mL^5} = 120\frac{EI}{mL^4}
\end{aligned}$$

We can imagine this may not be a good estimate of natural frequency. Because the second derivative of the shape function (1) is constant (= 2), which means that the curvature of the beam is uniform along the beam length.

Because the curvature should be zero at simply supported beam ends, let us assume the shape function (2) so that its second derivative becomes parabolic.

Shape function (2)

$$y(x) = x^4 - 2Lx^3 + L^3x \qquad y'(x) = 4x^3 - 6Lx^2 + L^3 \qquad y''(x) = 12(x^2 - Lx)$$

Then,

$$\begin{aligned}
k^* &= \int_0^L EI\{12(x^2 - Lx)\}^2 dx = 144EI\int_0^L (x^4 - 2Lx^3 + L^2x^2)dx \\
&= 144EI\Big[\frac{x^5}{5} - 2L\frac{x^4}{4} + L^2\frac{x^3}{3}\Big]_0^L = 144EIL^5\Big(\frac{1}{5} - \frac{1}{2} + \frac{1}{3}\Big) = \frac{24}{5}EIL^5
\end{aligned}$$

$$\begin{aligned}
m^* &= \int_0^L m\,(x^4 - 2Lx^3 + L^3x)^2 dx \\
&= m\int_0^L (x^8 + 4L^2x^6 + L^6x^2 - 4Lx^7 - 4L^4x^4 + 2L^3x^5)dx \\
&= m\Big[\frac{x^9}{9} + 4L^2\frac{x^7}{7} + L^6\frac{x^3}{3} - 4L\frac{x^8}{8} - 4L^4\frac{x^5}{5} + 2L^3\frac{x^6}{6}\Big]_0^L \\
&= mL^9\Big(\frac{1}{9} + \frac{4}{7} + \frac{1}{3} - \frac{1}{2} - \frac{4}{5} + \frac{1}{3}\Big) \\
&= mL^9\frac{1}{630}(70 + 360 + 210 - 315 - 504 + 210) = \frac{31}{630}mL^9 \\
\omega_{\mathrm{n}}^2 &= \frac{k^*}{m^*} = \frac{24}{5}EIL^5\frac{630}{31}\frac{1}{mL^9} = \frac{3024}{31}\frac{EI}{mL^4} \approx 97.5\frac{EI}{mL^4}
\end{aligned}$$

Now, let us assume the shape function (3) as follows.

Shape function (3)

$$y(x) = \sin\frac{\pi x}{L} \qquad y'(x) = \frac{\pi}{L}\cos\frac{\pi x}{L} \qquad y''(x) = -\frac{\pi^2}{L^2}\sin\frac{\pi x}{L}$$

$$\begin{aligned} k^* &= \int_0^L EI\Big(-\frac{\pi^2}{L^2}\sin\frac{\pi x}{L}\Big)^2 dx = EI\frac{\pi^4}{L^4}\int_0^L \frac{1}{2}\Big(1-\cos\frac{2\pi x}{L}\Big)dx \\ &= EI\frac{\pi^4}{2L^4}\Big[x - \frac{L}{2\pi}\sin\frac{2\pi x}{L}\Big]_0^L = EI\frac{\pi^4}{2L^4}\Big[L\Big] = \frac{\pi^4}{2}\frac{EI}{L^3} \end{aligned}$$

$$\begin{aligned} m^* &= m\int_0^L \sin^2\frac{\pi x}{L}\,dx = m\int_0^L \frac{1}{2}\big(1-\cos\frac{2\pi x}{L}\big)dx \\ &= \frac{m}{2}\Big[x - \frac{L}{2\pi}\sin\frac{2\pi x}{L}\Big]_0^L = \frac{mL}{2} \end{aligned}$$

$$\omega_{\mathrm{n}}^2 = \frac{k^*}{m^*} = \frac{\pi^4}{2}\frac{EI}{L^3}\frac{2}{mL} \approx 97.4\frac{EI}{mL^4}$$

This is the best estimate of the natural frequency, because the shape function (3) is the true shape function. It can be seen that the shape function (2) gives the natural frequency which is almost identical to the one that is derived using the shape function (3). This indicates that it is better to select a shape function which satisfies boundary conditions not only in terms of deflection but also in terms of curvature.

(3) Selection of Shape Function

The accuracy of the vibration frequency obtained by Rayleigh's method depends entirely on the shape function $\psi(x)$. In principle, any shape function may be selected which satisfies the geometric boundary conditions. However, any shape other than the true vibration shape would require the action of additional external constraints to maintain equilibrium; these extra constraints would stiffen the system and thus would cause an increase in the computed frequency. Consequently, the true vibration shape will yield the lowest frequency obtainable by Rayleigh's method and the lowest frequency given by this method is always the best approximation.

The correct vibration shape $\psi_{\mathrm{c}}(x)$ is the deflected shape that results from a loading $P_{\mathrm{c}}(x)$ proportional to $m(x)\psi_{\mathrm{c}}(x)$. It is not possible to guess the exact shape $\psi_{\mathrm{c}}(x)$ for a complex system, but the Rayleigh procedure will provide good accuracy with the deflection shape computed from the loading $\bar{P}(x) = m(x)\bar{\psi}(x)$, where $\bar{\psi}(x)$ is any reasonable approximation of the true shape. One common assumption is that the inertia force $\bar{P}(x)$ is merely the weight of the beam, that is, $\bar{P}(x) = m(x)g$. The maximum strain energy can then be found very simply, from the fact that the stored energy must be equal to the work done in the system by the applied loading.

$$E_{\mathrm{Pmax}} = \frac{1}{2}\int \bar{P}(x)\, y_{\mathrm{d}}(x) dx = \frac{1}{2} g y^* \int m(x)\, \bar{\psi}(x) dx$$

where $y_{\mathrm{d}}(x)$ is the deflected shape resulting from the dead load. The kinetic energy is given by Eq.(2.6.8b), in which $\bar{\psi}(x) = y_{\mathrm{d}}(x)/y^*$. Thus the frequency can be given by

$$\begin{aligned}\omega_{\mathrm{n}}^2 &= \frac{g}{y^*}\frac{\int m(x)\bar{\psi}(x)dx}{\int m(x)\{\bar{\psi}(x)\}^2 dx} \\ &= \frac{g\int m(x) y_{\mathrm{d}}(x) dx}{\int m(x)\{y_{\mathrm{d}}(x)\}^2 dx} \qquad (2.6.13)\end{aligned}$$

The loading $\bar{P}(x)$ is the gravitational loading in cases where the principal vibratory motion is in the vertical direction. For a structure like a vertical cantilever, the loading must be applied laterally, because the principal motion is horizontal. We must be cautioned against spending too much time in computing deflected shapes, for any reasonable shape assumption will give useful results.

(4) Improved Rayleigh's Method

The idea of using a deflected shape resulting from an inertia loading can be applied to improved versions of the procedure.

Method $\mathbf{R}_{00}$

Let us select the arbitrary shape function which satisfies the geometric boundary conditions. We have

$$y^{(0)}(x,t) = \psi^{(0)}\, y^* \sin\omega_{\mathrm{n}} t \qquad (2.6.14)$$

where superscript zero denotes the initial values. The maximum potential and kinetic energy are given by

$$\begin{aligned}E_{\mathrm{Pmax}} &= \frac{1}{2}\int EI(x)\{\frac{\partial^2 y^{(0)}}{\partial x^2}\}^2 dx \\ &= \frac{y^{*(0)^2}}{2}\int EI(x)\{\psi''^{(0)}\}^2 dx \qquad (2.6.15)\end{aligned}$$

$$\begin{aligned}E_{\mathrm{Kmax}} &= \frac{1}{2}\int m(x)\{\dot{y}^{(0)}\}^2 dx \\ &= \frac{y^{*(0)^2}}{2}\omega_{\mathrm{n}}^2\int m(x)\{\psi^{(0)}\}^2 dx \qquad (2.6.16)\end{aligned}$$

The standard Rayleigh frequency expression, designated as R_{00}, is

$$\omega_{\mathrm{n}}^2 = \frac{\int EI(x)\{\psi''^{(0)}\}^2 dx}{\int m(x)\{\psi^{(0)}\}^2 dx} \qquad (2.6.17)$$

Method $\mathbf{R}_{01}$

A better approximation of the frequency can be obtained by computing the potential energy from the work done in deflecting the system by the inertia force associated with the assumed deflection. The inertia force at the time of maximum displacement is,

$$p^{(0)}(x) = \omega_{\mathrm{n}}^2 m(x)\, y^{(0)} = y^{*(0)}\omega_{\mathrm{n}}^2 m(x)\, \psi^{(0)} \tag{2.6.18}$$

The deflection caused by this loading may be expressed as,

$$y^{(1)} = \omega_{\mathrm{n}}^2 \frac{y^{(1)}}{\omega_{\mathrm{n}}^2} = \omega_{\mathrm{n}}^2\, \psi^{(1)} \frac{y^{*(1)}}{\omega_{\mathrm{n}}^2} = \omega_{\mathrm{n}}^2\, \psi^{(1)}\, \bar{y}^{*(1)} \tag{2.6.19}$$

where ω_{n} is the unknown frequency. The potential energy of the strain produced by this loading is given by,

$$\begin{aligned} E_{\mathrm{Pmax}} &= \frac{1}{2}\int p^{(0)}(x)\, y^{(1)} dx \\ &= \frac{1}{2} y^{*(0)}\, \bar{y}^{*(1)} \omega_{\mathrm{n}}^4 \int m(x)\, \psi^{(0)}\, \psi^{(1)} dx \end{aligned} \tag{2.6.20}$$

Equating this to the kinetic energy given by the originally assumed shape [Eq.(2.6.16)] leads to the improved Rayleigh frequency expression, R_{01}:

$$\omega_{\mathrm{n}}^2 = \frac{y^{*(0)}}{\bar{y}^{*(1)}} \frac{\int m(x)\{\psi^{(0)}\}^2 dx}{\int m(x)\, \psi^{(0)}\, \psi^{(1)}\, dx} \tag{2.6.21}$$

This is often recommended in preference to Eq.(2.6.17) because it avoids the differentiation operation and will give improved accuracy.

Method $\mathbf{R}_{11}$

A still better approximation can be obtained with relatively additional effort by computing the kinetic energy from the calculated shape $y^{(1)}$. In this case,

$$\begin{aligned} E_{\mathrm{Kmax}} &= \frac{1}{2}\int m(x)\{\dot{y}^{(1)}\}^2 dx \\ &= \frac{\omega_{\mathrm{n}}^6}{2}\{\bar{y}^{*(1)}\}^2 \int m(x)\{\psi^{(1)}\}^2 dx \end{aligned} \tag{2.6.22}$$

Equating this to Eq.(2.6.20) leads to the further improved result (R_{11} method):

$$\omega_{\mathrm{n}}^2 = \frac{y^{*(0)}}{\bar{y}^{*(1)}} \frac{\int m(x)\, \psi^{(0)}\, \psi^{(1)} dx}{\int m(x)\{\psi^{(1)}\}^2 dx} \tag{2.6.23}$$

Further improvement could be obtained by continuing the process another step, that is, by using the inertia loading associated with $\psi^{(1)}$ to calculate a new shape $\psi^{(2)}$. In fact, the process will eventually converge with the exact vibration shape. However, for practical use of Rayleigh's method there is no need to go beyond the improved procedure represented by Eq.(2.6.23).

[**Example 2.6**]

A concentrated mass M is attached to the center of a simply supported uniform beam, whose length is L, mass per unit length is m and flexural stiffness is EI, as shown in Fig. E2.6. Calculate the fundamental natural period of the beam, using Rayleigh's method with the prescribed shape function.

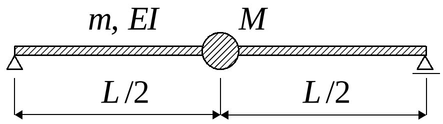


Fig.E2.6

Where, $L = 100$ (cm), $m = 2.0 \times 10^{-5}$ (kN s^2/cm^2), $M = 0.1$ (kN s^2/cm), $E = 2\,100$ (kN /cm^2), $I = 6.0 \times 10^5$ (cm^4)

Shape function (1) : $y(x) = \sin\left(\frac{\pi x}{L}\right)$

Shape function (2) : $y(x) = x\left(3L^2 - 4x^2\right)$ for $\left(x \leq \frac{L}{2}\right)$

[**Solution**]

For shape function (1)

$$y(x) = \sin\left(\frac{\pi x}{L}\right) \qquad y'(x) = \frac{\pi}{L}\cos\left(\frac{\pi x}{L}\right) \qquad y''(x) = -\frac{\pi^2}{L^2}\sin\left(\frac{\pi x}{L}\right)$$

$$\begin{aligned}
k^* &= \int_0^L EI\left\{y''(x)\right\}^2 dx = EI\frac{\pi^4}{L^4}\int_0^L \left(\sin\frac{\pi x}{L}\right)^2 dx \\
&= EI\frac{\pi^4}{L^4}\int_0^L \frac{1}{2}\left(1 - \cos\frac{2\pi x}{L}\right)dx = EI\frac{\pi^4}{L^4}\left[\frac{x}{2} - \frac{L}{4\pi}\sin\frac{2\pi x}{L}\right]_0^L \\
&= EI\frac{\pi^4}{L^4}\left[\frac{L}{2}\right] = \frac{EI}{2L^3}\pi^4 \\
&= \frac{2100 \times 6 \times 10^5}{2 \times 100^3} \times \pi^4 = 630\pi^4 \approx 61370
\end{aligned}$$

$$\begin{aligned}
m^* &= \int_0^L m\,y^2 dx + M\left\{y\left(x = \frac{L}{2}\right)\right\}^2 = m\int_0^L \sin^2\frac{\pi x}{L}\,dx + M\left(\sin\frac{\pi}{L}\frac{L}{2}\right)^2 \\
&= m\int_0^L \frac{1}{2}\left(1 - \cos\frac{2\pi x}{L}\right)dx + M = m\left[\frac{x}{2} - \frac{L}{4\pi}\sin\frac{2\pi x}{L}\right]_0^L + M \\
&= \frac{Lm}{2} + M = \frac{100}{2} \times 2.0 \times 10^{-5} + 0.1 = 0.101
\end{aligned}$$

$$\omega_{\mathrm{n}}^2 = \frac{k^*}{m^*} \approx \frac{61370}{0.101} \approx 6.076 \times 10^5 \qquad \omega_{\mathrm{n}} = 779$$

For shape function (2)

$$y(x) = x\left(3L^2 - 4x^2\right) \qquad y'(x) = 3L^2 - 12x^2 \qquad y''(x) = -24x$$

$$
\begin{aligned}
k^* &= \int_0^L EI\{y''(x)\}^2 dx = 2EI\int_0^{L/2}(-24x)^2 dx = 1152\,EI\Big[\frac{x^3}{3}\Big]_0^{L/2} = 1152\,EI\Big[\frac{1}{3}\frac{L^3}{2^3}\Big] \\
&= 48EIL^3 = 48\times 2100\times 6\times 10^5\times 100^3 = 6.048\times 10^{16}
\end{aligned}
$$

$$
\begin{aligned}
m^* &= \int_0^L m\,y^2 dx + M\{y(x=\frac{L}{2})\}^2 \\
&= 2m\int_0^{L/2}\{x(3L^2-4x^2)\}^2\,dx + M\{\frac{L}{2}(3L^2-4\frac{L^2}{2^2})\}^2 \\
&= 2m\int_0^{L/2}(9L^4x^2-24L^2x^4+16x^6)dx + ML^6 \\
&= 2m\Big[\frac{9}{3}L^4x^3-\frac{24}{5}L^2x^5+\frac{16}{7}x^7\Big]_0^{L/2} + ML^6 = 2mL^7\Big[\frac{3}{8}-\frac{24}{5\times 32}+\frac{16}{7\times 2^7}\Big]+ML^6 \\
&= mL^7\Big[\frac{3}{4}-\frac{3}{10}+\frac{1}{28}\Big]+ML^6 = mL^7\Big[\frac{105}{140}-\frac{42}{140}+\frac{5}{140}\Big]+ML^6 \\
&= \frac{17}{35}mL^7+ML^6 = \frac{17}{35}2\times 10^{-5}\times 100^7+0.1\times 100^6 = 0.97\times 10^9+0.1\times 10^{12} \\
&\approx 0.101\times 10^{12}
\end{aligned}
$$

$$
\omega_{\mathrm{n}}^2 = \frac{k^*}{m^*} \approx \frac{6.048\times 10^{16}}{0.101\times 10^{12}} \approx 5.988\times 10^5 \qquad \omega_{\mathrm{n}} = 774
$$

The frequency obtained from the shape function (2), $\omega_{\mathrm{n}} = 774$, is smaller than that obtained from the shape function (1), $\omega_{\mathrm{n}} = 779$. Since the lesser frequency obtained by Rayleigh's method always gives better approximation, the shape function (2) is closer to the real mode shape.

2.7 Frequency Domain Analysis

Let us remember Fourier transformation. This is a fundamental mathematical technique with which an ordinary function can be decomposed into many simple harmonic motions with different frequencies. When the external force $p(t)$ is expanded into many terms of simple harmonic motions by Fourier transformation, the response to each term can be obtained by the procedure in Section 2.3 for the solution of simple harmonic loading. When the solution for each harmonic loading is obtained, the solution for $p(t)$ is then the sum of all those responses.

(1) Complex Frequency Response Function

An equation of motion for a damped SDOF system subjected to the external force $p(t)$ is

$$
m\,\ddot{x}(t) + c\,\dot{x}(t) + k\,x(t) = p(t)
$$

Then consider the external force that is

$$
p(t) = p_0\,e^{i\omega t} \tag{2.7.1}
$$

Remembering the Euler's formula, $e^{i\omega t} = \cos\omega t + i\sin\omega t$, the above equation represents sunusoidal and cosine forces together. Then the equation of motion becomes

$$m\,\ddot{x}(t) + c\,\dot{x}(t) + k\,x(t) = p_0\,e^{i\omega t} \tag{2.7.2}$$

Since the external force is harmonic, let us assume that the solution is also hamonic with the same frequency as the external force.

$$x(t) = p_0\,H(\omega)\,e^{i\omega t} \tag{2.7.3a}$$

where $H(\omega)$ remains to be determined. Differenciating the above equation, we obtain

$$\dot{x}(t) = i\omega p_0 H(\omega) e^{i\omega t} \qquad\qquad \ddot{x}(t) = -\omega^2 p_0 H(\omega) e^{i\omega t} \tag{2.7.3b}$$

Substituting these equations into Eq.(2.7.2), we have

$$m\{-\omega^2 p_0 H(\omega) e^{i\omega t}\} + c\{i\omega p_0 H(\omega) e^{i\omega t}\} + k\{p_0 H(\omega) e^{i\omega t}\} = p_0\,e^{i\omega t}$$

$$\{-\omega^2 m + i\omega c + k\} p_0 H(\omega)\,e^{i\omega t} = p_0\,e^{i\omega t}$$

Then we have

$$\begin{aligned} H(\omega) &= \frac{1}{-\omega^2 m + i\omega c + k} \\ &= \frac{1}{k}\frac{1}{\{1-(\omega/\omega_{\rm n})^2\} + i\{2\zeta(\omega/\omega_{\rm n})\}} \\ &= \frac{1}{k}\frac{1}{(1-r_{\rm n}^2) + i(2\zeta r_{\rm n})} \end{aligned} \tag{2.7.4}$$

where, as already defined, $\omega_{\rm n}^2 = k/m$, $\zeta = c/(2m\omega_{\rm n}) = c\,\omega_{\rm n}/(2k)$ and $r_{\rm n} = \omega/\omega_{\rm n}$.

$H(\omega)$ is known as the complex frequency response function and represents the steady state response of the system to a harmonic force of unit amplitude.

(2) Response to Arbitrary Excitation

Any excitation $p(t)$ can be represented by the Fourier integral as follows.

$$p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(\omega) e^{i\omega t} d\omega \tag{2.7.5a}$$

where,

$$P(\omega) = \int_{-\infty}^{\infty} p(t) e^{-i\omega t} dt \tag{2.7.5b}$$

Eq.(2.7.5b) represents the Fourier transform of the time function of $p(t)$, and Eq.(2.7.5a) is the inverse Fourier transform of the frequency function $P(\omega)$.

The response of a linear sysytem to excitation $p(t)$ can be obtained by combining the response to each harmonic excitation term of Eq.(2.7.5a). Then,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{i\omega t} d\omega \tag{2.7.6a}$$

$$X(\omega) = H(\omega)P(\omega) \tag{2.7.6b}$$

(3) Complex Frequency Response Function and Unit Impulse Function

The impulse response function of Eq.(2.4.14) can be written for $\tau = 0$ as

$$h(t) = \frac{1}{m\omega_{\mathrm{d}}} e^{-\zeta\omega_{\mathrm{n}} t} \sin \omega_{\mathrm{d}} t \tag{2.7.7}$$

Substituting $p(t) = \delta(t)$ into Eq.(2.7.5b), the Fourier transform for the unit impulse is given as follows.

$$P(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-i\omega t} dt \tag{2.7.8}$$

Substituting $P(\omega) = 1$ into Eq.(2.7.6b), Eq.(2.7.6a) gives

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{i\omega t} d\omega \tag{2.7.9a}$$

We can see that $h(t)$ is the inverse Fourier transform of $H(\omega)$, and $H(\omega)$ is the Fourier transform of $h(t)$.

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt \tag{2.7.9b}$$

(4) Discrete Fourier Transform

The frequency domain analysis requires that the Fourier transform of $p(t)$, Eq.(2.7.5b), and the inverse Fourier transform of $X(\omega)$, Eq.(2.7.6a), be evaluated numerically and the numerical evaluation requires truncating these integrals over an infinite range to a finite range. This is equivalent to approximating the arbitrary time-varing excitation by a periodic function.

[Finite complex Fourier series]

When the arbitrary loading is expressed as N digitized values p_m,

$$p_m = \sum_{k=0}^{N/2-1} P_k e^{i(2\pi km/N)} \qquad m = 0, 1, 2, \ldots, N-1 \tag{2.7.10a}$$

$$P_k = \frac{1}{N} \sum_{k=0}^{N/2-1} p_m e^{-i(2\pi km/N)} \qquad k = 0, 1, 2, \ldots, N-1 \tag{2.7.10b}$$

The response will be,

$$x_{(t=t_m)} = x_m = \sum_{k=0}^{N/2-1} H(\tfrac{2\pi k}{N\Delta t})\, P_k\, e^{i(2\pi km/N)} \qquad m = 0, 1, 2, \ldots, N-1 \tag{2.7.11}$$

[Finite Fourier series]

Complex algebra is required to calculate equations for the finite complex Fourier series. The calculation without complex algebra can be done using the finite Fourier series as follows:

$$p_m = \frac{A_0}{2} + \sum_{k=0}^{N/2-1} (A_k \cos \omega_k t + B_k \sin \omega_k t) + \frac{A_{N/2}}{2} \cos \omega_{N/2} t \tag{2.7.12}$$

where, $\omega_k = \dfrac{2\pi k}{N\Delta t}$

$$A_k = \frac{2}{N} \sum_{m=0}^{N-1} p_m \cos \frac{2\pi k m}{N} \qquad k = 0, 1, 2, \ldots, N/2-1, N/2 \tag{2.7.13a}$$

$$B_k = \frac{2}{N} \sum_{m=0}^{N-1} p_m \sin \frac{2\pi k m}{N} \qquad k = 1, 2, \ldots, N/2-1 \tag{2.7.13b}$$

The coefficient of Eq.(2.7.10a) and the above coefficients have the relationship as follows:

$$P_k = \frac{A_k - iB_k}{2}, \qquad 0 \le k \le N/2 \tag{2.7.14a}$$

$$P_{N-k} = \frac{A_k - iB_k}{2}, \qquad 1 \le k \le N/2-1 \tag{2.7.14b}$$

The response will be,

$$x(t = t_m) = x_m = \sum_{k=0}^{N/2-1} R_{\mathrm{dy}}\, A_k \cos \frac{2\pi k m}{N} + \sum_{k=1}^{N/2-1} R_{\mathrm{dy}}\, B_k \sin \frac{2\pi k m}{N} \qquad m = 0, 1, 2, \ldots, N-1 \tag{2.7.15}$$

where the dynamic magnification factor R_{dy} has been already given by Eq.(2.3.21).

Chapter 3

Multi Degree of Freedom (MDOF) Systems

3.1 Equations of Motion

Equations of motion for a two-degree of freedom system, as illustrated in Fig.3.1.1, can be derived as follows.

Considering the equilibrium of the forces acting on the first mass, we have

$$-m_1\,\ddot{x}_1 - c_1\,\dot{x}_1 + c_2(\dot{x}_2 - \dot{x}_1) - k_1\,x_1 + k_2(x_2 - x_1) + p_1 = 0 \tag{3.1.1a}$$

Similarly as to the second mass, we have

$$-m_2\,\ddot{x}_2 - c_2(\dot{x}_2 - \dot{x}_1) - k_2(x_2 - x_1) + p_2 = 0 \tag{3.1.1b}$$

It should be noted that for simplicity, from this section on, x and p are used instead of $x(t)$ and $p(t)$, respectively. Rearranging the above equations yields

$$m_1\,\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\,\dot{x}_2 + (k_1 + k_2)x_1 - k_2\,x_2 = p_1 \tag{3.1.2a}$$

$$m_2\,\ddot{x}_2 - c_2\,\dot{x}_1 + c_2\,\dot{x}_2 - k_2\,x_1 + k_2\,x_2 = p_2 \tag{3.1.2b}$$

When we rewrite these equations in a matrix form, we have

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} \tag{3.1.3a}$$

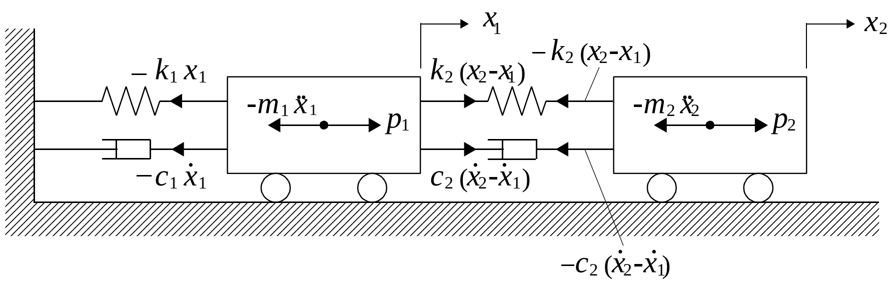


Fig.3.1.1 Two-degree of freedom system

or

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = \{p\} \tag{3.1.3b}$$

where $[m]$, $[c]$ and $[k]$ are called the mass matrix, damping matrix and stiffness matrix, respectively, and $\{\ddot{x}\}$, $\{\dot{x}\}$, $\{x\}$ and $\{p\}$ are the acceleration vector, velocity vector, displacement vector and force vector, respectively.

It can be seen that Eq.(3.1.3b) corresponds exactly to Eq.(2.1.1) which is the equation of motion of a SDOF system, except that the former has a matrix form. When we calculate the multiplication of the third term of Eq.(3.1.3b), we obtain the forces $p_{\mathrm{s}1}$ and $p_{\mathrm{s}2}$ which are caused by the springs and the displacements.

$$p_{\mathrm{s}1} = k_{11}\,x_1 + k_{12}\,x_2 \tag{3.1.4a}$$

$$p_{\mathrm{s}2} = k_{21}\,x_1 + k_{22}\,x_2 \tag{3.1.4b}$$

The element k_{11} is the force produced at the 1st mass due to a unit displacement at the 1st mass only, and k_{12} is the force produced at the 1st mass due to a unit displacement at the 2nd mass only, etc. Therefore, each element of the matrices has the following physical meaning: m_{ij}, c_{ij}, and k_{ij} are the forces at the i-th mass due to a unit of acceleration, velocity and displacement applied at the j-th mass, respectively, with a restriction that all other accelerations, velocities and displacements are equal to zero.

Then it can be seen that the first column of the stiffness matrix shows the forces applied at each mass to give a unit displacement at the first mass, etc. Similarly, the i-th column of the damping matrix shows the forces applied at each mass to give a unit velocity at i-th mass, etc. Furthermore we can see that $m_{ij} = m_{ji}$, $c_{ij} = c_{ji}$ and $k_{ij} = k_{ji}$ by Betti's law, i.e. the work done by one set of loads on the deflections due to a second set of loads is equal to the work of the second set of loads acting on the deflections due to the first.

Let us derive the equation of motion of a rather complicated system in Fig.3.1.2. The displacement vector can be defined as follows.

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ \theta \\ x_{\mathrm{s}} \end{Bmatrix} \tag{3.1.5}$$

where x_1, x_2, θ and x_{s} are the horizontal displacement of m_1, the horizontal displacement of m_2, the rotation angle of I_θ and the horizontal displacement of M, respectively. The mass matrix is

$$[m] = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & I_\theta & 0 \\ 0 & 0 & 0 & M \end{bmatrix} \tag{3.1.6}$$

The elements of the first column of the stiffness matrix are the forces that give a unit

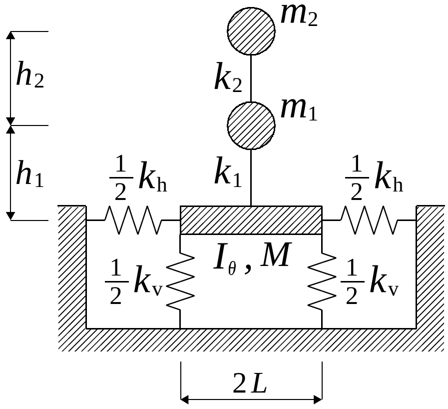


Fig.3.1.2 Model of MDOF systems

displacement at m_1 (see Fig.3.1.3a). Then,

$$k_{11} = k_1 + k_2 \tag{3.1.7a}$$
$$k_{21} = -k_2 \tag{3.1.7b}$$
$$k_{31} = k_1\, h_1 - k_2\, h_2 \tag{3.1.7c}$$
$$k_{41} = -k_1 \tag{3.1.7d}$$

The elements of the second column are (see Fig.3.1.3b)

$$k_{12} = -k_2 \tag{3.1.8a}$$
$$k_{22} = k_2 \tag{3.1.8b}$$
$$k_{32} = k_2\, h_2 \tag{3.1.8c}$$
$$k_{42} = 0 \tag{3.1.8d}$$

The elements of the third column are (see Fig.3.1.3c)

$$k_{13} = k_1\, h_1 - k_2\, h_2 \tag{3.1.9a}$$
$$k_{23} = k_2\, h_2 \tag{3.1.9b}$$
$$k_{33} = k_\theta + k_1\, h_1^2 + k_2\, h_2^2 \tag{3.1.9c}$$
$$k_{43} = -k_1\, h_1 \tag{3.1.9d}$$

where, $k_\theta = L^2\, k_\mathrm{v}$.

The elements of the fourth column are (see Fig.3.1.3d)

$$k_{14} = -k_1 \tag{3.1.10a}$$
$$k_{24} = 0 \tag{3.1.10b}$$
$$k_{34} = -k_1\, h_1 \tag{3.1.10c}$$
$$k_{44} = k_h + k_1 \tag{3.1.10d}$$

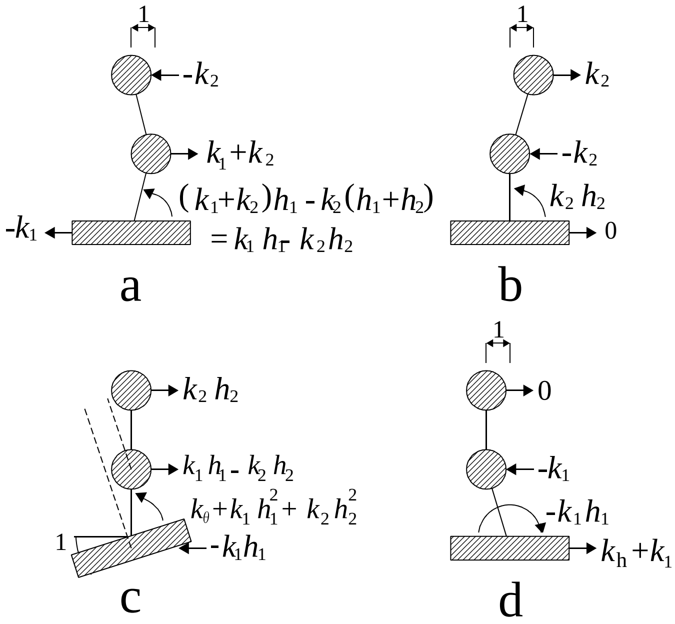


Fig.3.1.3 Elements of stiffness matrix

[**Example 3.1**]

Derive the equation of motion of the system subjected to the ground acceleration $\ddot{x}_g$ as shown in Fig.E3.1, where $m_1 = m_2 = m$, $c_1 = 3c$, $c_2 = 2c$ and $k_1 = 3k$, $k_2 = 2k$.

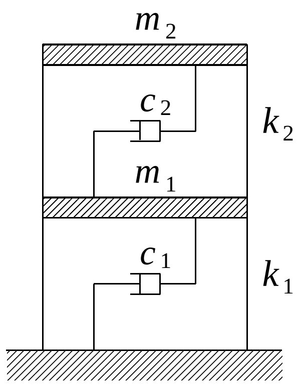


Fig.E3.1

[**Solution**]

The mass matrix is

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

The damping matrix is

$$[c] = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} = \begin{bmatrix} 5c & -2c \\ -2c & 2c \end{bmatrix}$$

The stiffness matrix is

$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} 5k & -2k \\ -2k & 2k \end{bmatrix}$$

Therefore the equation of motion is

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 5c & -2c \\ -2c & 2c \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 5k & -2k \\ -2k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = -\ddot{x}_g \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

where, $\{1\}$ is a vector of ones.

In general, we can write the equation of motion for a MDOF system subjected to the ground acceleration as follows.

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = -\ddot{x}_{\mathrm{g}}[m]\{1\}$$

3.2 Undamped Free Vibration

Omitting the damping forces and external forces from Eq.(3.1.3b), the equation of motion of undamped free vibration is given by

$$[m]\{\ddot{x}\} + [k]\{x\} = \{0\} \tag{3.2.1}$$

When we assume that the motion for free vibration is a simple harmonic, as we did in the case of the SDOF system, the displacement vector can be written in the form of

$$\{x\} = \{u\}\sin(\omega_j t + \theta) \tag{3.2.2}$$

where $\{u\}$ is the shape of a system that does not change with respect to time and θ is the phase angle. The second derivative of this equation can be obtained as

$$\{\ddot{x}\} = -\omega_j^2\{u\}\sin(\omega_j t + \theta) \tag{3.2.3}$$

Substituting the above two equations into Eq.(3.2.1) yields

$$-\omega_j^2[m]\{u\}\sin(\omega_j t + \theta) + [k]\{u\}\sin(\omega_j t + \theta) = \{0\}$$

Then,

$$\big([k] - \omega_j^2[m]\big)\{u\}\sin(\omega_j t + \theta) = \{0\}$$

Since the sine term changes with time, the following relation must be satisfied.

$$\big([k] - \omega_j^2[m]\big)\{u\} = \{0\} \tag{3.2.4}$$

In order for a nontrivial solution of $\{u\}$ to be possible, the determinant of $\big([k] - \omega_j^2[m]\big)$ must be zero. Then,

$$\big|[k] - \omega_j^2[m]\big| = 0 \tag{3.2.5}$$

This equation is called the frequency equation (or characteristic equation) of the system. Expanding the determinant, this becomes the algebraic equation of the n degrees in the frequency parameter ω_j^2 for a system having n degree of freedom. The n roots of this equation represent the frequencies of the n modes of vibration which are possible in the system. The mode having the lowest frequency is called the first or fundamental mode, the second lowest is called the second mode, etc. The vector of the entire set of modal frequencies, arranged in sequence, is called the frequency vector $\{\omega_j\}$.

$$\{\omega_j\} = \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{Bmatrix} \tag{3.2.6}$$

When the frequencies of vibration are determined from Eq.(3.2.5), Eq.(3.2.4) becomes

$$[E_j]\{u\} = \{0\} \tag{3.2.7}$$

where,

$$[E_j] = [k] - {\omega_j}^2[m] \tag{3.2.8}$$

since $[E_j]$ depends on the frequency, it is different for each mode. The amplitude vector $\{u\}$ of the vibration is indeterminate, but the shape of the system can be determined. The procedure to determine the frequency and the corresponding amplitude shape, which is called the eigenvalue problem in mathematics, will be discussed later. If we let the vector be dimensionless by dividing all components by one reference component (usually the first or the largest), the vector can be determined, and it is called the j-th mode shape $\{\phi_j\}$. Where,

$$\{\phi_j\} = \begin{Bmatrix} \phi_{1j} \\ \phi_{2j} \\ \vdots \\ \phi_{nj} \end{Bmatrix} \tag{3.2.9}$$

The square matrix made up of the N mode shapes is called the mode shape matrix $[\phi]$.

$$[\phi] = \big[\{\phi_1\}\{\phi_2\}\cdots\{\phi_n\}\big] = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{bmatrix} \tag{3.2.10}$$

The preceding discussion of the equation of motion was based on the stiffness matrix. However, in many cases it is more convenient to express the elastic properties of the structural system by means of the flexibility matrix rather than the stiffness matrix. The flexibility matrix $[f]$ is the inverse of the stiffness matrix $[k]$.

$$[f] = [k]^{-1} \tag{3.2.11}$$

Eqs.(3.1.3a) and (3.1.3b) can be written in the form of

$$\{p_{\mathrm{s}}\} = [k]\{x\} \tag{3.2.12}$$

Substituting Eq.(3.2.11) to this equation yields

$$\{x\} = [f]\{p_{\mathrm{s}}\} \tag{3.2.13}$$

or

$$x_1 = f_{11}\, p_{\mathrm{s}1} + f_{12}\, p_{\mathrm{s}2} \tag{3.2.13a}$$
$$x_2 = f_{21}\, p_{\mathrm{s}1} + f_{22}\, p_{\mathrm{s}2} \tag{3.2.13b}$$

From these equations, it can be seen that the element of the flexibility matrix f_{ij} is the deflection at the i-th mass due to a unit force applied to the j-th mass. If we use the flexibility matrix, Eq.(3.2.5) can be written in the form of

$$\left| \frac{1}{\omega_j^2}[I] - [f][m] \right| = 0 \tag{3.2.14}$$

where $[I]$ represents an identity matrix of order n.

The flexibility matrix is given as the inverse matrix of the stiffness matrix, i.e. $[f] = [k]^{-1}$. In case the flexibility matrix is given, the stiffness matrix can be obtained as the inverse matrix of the flexibility matrix, i.e. $[k] = [f]^{-1}$.

[Example 3.2]

Find the frequency vector and mode shape matrix of the system as shown in Fig.E3.2, where $m_1 = m_2 = m$ and $k_1 = 3k$, $k_2 = 2k$.

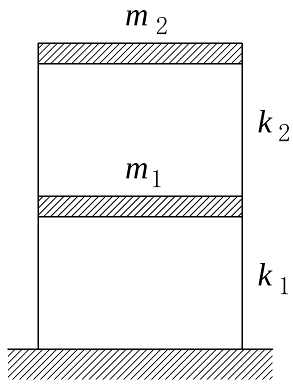


Fig.E3.2

[Solution]

The mass matrix is

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

The stiffness matrix is

$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} 5k & -2k \\ -2k & 2k \end{bmatrix}$$

The frequency equation becomes

$$\left| [k] - \omega_n^2 [m] \right| = \left| \begin{bmatrix} 5k & -2k \\ -2k & 2k \end{bmatrix} - \omega_n^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 5k - m\omega_j^2 & -2k \\ -2k & 2k - m\omega_j^2 \end{vmatrix} = 0$$

$$(5k - m\omega_j^2)(2k - m\omega_j^2) - (-2k)(-2k) = 0$$

$$(m\omega_j^2)^2 - 7k(m\omega_j^2) + 6k^2 = 0$$

$$(m\omega_j^2 - k)(m\omega_j^2 - 6k) = 0$$

$$m\omega_j^2 = k \qquad \text{or} \qquad m\omega_j^2 = 6k$$

$$\omega_j^2 = \frac{k}{m} \qquad \text{or} \qquad \omega_j^2 = \frac{6k}{m}$$

$$\omega_j = \sqrt{\frac{k}{m}} \qquad \text{or} \qquad \omega_j = \sqrt{\frac{6k}{m}}$$

Then the frequency vector is given as

$$\{\omega\} = \begin{Bmatrix} \sqrt{\frac{k}{m}} \\ \sqrt{\frac{6k}{m}} \end{Bmatrix} = \sqrt{\frac{k}{m}} \begin{Bmatrix} 1 \\ \sqrt{6} \end{Bmatrix}$$

When $\omega_1 = \sqrt{k/m}$, Eq.(3.2.4) becomes,

$$\begin{bmatrix} 4k & -2k \\ -2k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Then,

$$u_2 = 2u_1$$

When $\omega_2 = \sqrt{6k/m}$, Eq.(3.2.4) becomes,

$$\begin{bmatrix} -k & -2k \\ -2k & -4k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Then,

$$u_2 = -\frac{1}{2}u_1$$

Then the mode shape matrix is given as

$$[\phi] = \begin{bmatrix} 1 & 1 \\ 2 & -\frac{1}{2} \end{bmatrix}$$

3.3 Orthogonality Conditions

The mode shape for an undamped system in free vibration of the j-th mode can be written as follows [see Eq.(3.2.4)].

$$[k]\{u_j\} = \omega_j^2[m]\{u_j\} \tag{3.3.1a}$$

This equation must be satisfied for all mode vectors. Then we have for the k-th mode

$$[k]\{u_k\} = \omega_k^2[m]\{u_k\} \tag{3.3.1b}$$

Taking the transposition of both sides of Eq.(3.3.1a) and post-multiplying both sides by $\{u_k\}$, we have

$$([k]\{u_j\})^{\mathrm{T}}\{u_k\} = \omega_j^2([m]\{u_j\})^{\mathrm{T}}\{u_k\} \tag{3.3.2}$$

Remembering the relationships $\big([A][B]\big)^{\mathrm{T}} = [B]^{\mathrm{T}}[A]^{\mathrm{T}}$, and the matrices $[k]$ and $[m]$ are symmetrical, i.e. $[k]^{\mathrm{T}} = [k]$ and $[m]^{\mathrm{T}} = [m]$, this equation becomes

$$\{u_j\}^{\mathrm{T}}[k]\{u_k\} = \omega_j^2\{u_j\}^{\mathrm{T}}[m]\{u_k\} \tag{3.3.3a}$$

Pre-multiplying both sides of Eq.(3.3.1b) by $\{u_j\}^{\mathrm{T}}$, we have

$$\{u_j\}^{\mathrm{T}}[k]\{u_k\} = \omega_k^2\{u_j\}^{\mathrm{T}}[m]\{u_k\} \tag{3.3.3b}$$

This equation is subtracted from each side of Eq.(3.3.3a).

$$(\omega_j^2 - \omega_k^2)\{u_j\}^{\mathrm{T}}[m]\{u_k\} = 0$$

Apparently, $\omega_j \neq \omega_k$. Therefore, we have for different mode shapes

$$\{u_j\}^{\mathrm{T}}[m]\{u_k\} = 0 \quad \text{for} \quad \omega_j \neq \omega_k \tag{3.3.4}$$

This is the first orthogonality of eigenvectors, i.e. the different mode shapes are orthogonal with respect to the mass matrix. This means that the off-diagonal elements of the product $[u]^{\mathrm{T}}[m][u]$ are zero.

The second orthogonality can be given from this relation. Pre-multiplying Eq.(3.3.1b) by $\{u_j\}^{\mathrm{T}}$ and using Eq.(3.3.4), we have

$$\{u_j\}^{\mathrm{T}}[k]\{u_k\} = 0 \quad \text{for} \quad \omega_j \neq \omega_k \tag{3.3.5}$$

This equation shows that the different mode shapes are orthogonal with respect to stiffness matrix. The previous two equations indicate that the mass matrix and stiffness matrix are diagonalized by the same matrix $[u]$.

Pre-multiplying both sides of Eq.(3.3.1b) by $\{u_j\}^{\mathrm{T}}[k][m]^{-1}$ leads to

$$\{u_j\}^{\mathrm{T}}[k][m]^{-1}[k]\{u_k\} = \omega_k^2\{u_j\}^{\mathrm{T}}[k]\{u_k\}$$

from which (using Eq.(3.3.5))

$$\{u_j\}^{\mathrm{T}}[k][m]^{-1}[k]\{u_k\} = 0 \tag{3.3.6}$$

Pre-multiplying Eq.(3.3.1b) by $\{u_j\}^{\mathrm{T}}[k][m]^{-1}[k][m]^{-1}$ leads to

$$\{u_j\}^{\mathrm{T}}[k][m]^{-1}[k][m]^{-1}[k]\{u_k\} = \omega_k^2\{u_j\}^{\mathrm{T}}[k][m]^{-1}[k]\{u_k\}$$

from which (using Eq.(3.3.6))

$$\{u_j\}^{\mathrm{T}}[k][m]^{-1}[k][m]^{-1}[k]\{u_k\} = 0 \tag{3.3.7}$$

Pre-multiplying both sides of Eq.(3.3.1b) by $\frac{1}{\omega_j^2}\{u_j\}^{\mathrm{T}}[m][k]^{-1}$ leads to

$$\frac{1}{\omega_j^2}\{u_j\}^{\mathrm{T}}[m]\{u_k\} = \{u_j\}^{\mathrm{T}}[m][k]^{-1}[m]\{u_k\}$$

from which (using Eq.(3.3.4))

$$\{u_j\}^{\mathrm{T}}[m][k]^{-1}[m]\{u_k\} = 0 \tag{3.3.8}$$

Similarly we have

$$\{u_j\}^{\mathrm{T}}[m][k]^{-1}[m][k]^{-1}[m]\{u_k\} = 0 \tag{3.3.9a}$$

or, introducing the relation of $[k]^{-1} = [f]$

$$\{u_j\}^{\mathrm{T}}[m][f][m][f][m]\{u_k\} = 0 \tag{3.3.9b}$$

The above mentioned orthogonality conditions can be expressed as

$$\{u_j\}^{\mathrm{T}}[m]\big([m]^{-1}[k]\big)^b\{u_k\} = 0 \quad \text{for} \quad -\infty < b < \infty \tag{3.3.10a}$$

It should be noted that Eqs.(3.3.4) and (3.3.5) are given by this equation when $b = 0$ and $b = 1$, respectively.

Using the mode shape vector form, we have

$$\{\phi_j\}^{\mathrm{T}}[m]\big([m]^{-1}[k]\big)^b\{\phi_k\} = 0 \quad \text{for} \quad -\infty < b < \infty \tag{3.3.10b}$$

[**Example 3.3**]
For the system in Example 3.2, confirm the orthogonality conditions of Eqs.(3.3.4) and (3.3.5).

[**Solution**]
The mass matrix, stiffness matrix and mode shape matrix have been given as follows.

$$[m] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \qquad [k] = \begin{bmatrix} 5k & -2k \\ -2k & 2k \end{bmatrix} \qquad [\phi] = \begin{bmatrix} 1 & 1 \\ 2 & -\frac{1}{2} \end{bmatrix}$$

Therefore,

$$\{\phi_1\}^{\mathrm{T}}[m]\{\phi_2\} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ -\frac{1}{2} \end{Bmatrix} = \begin{Bmatrix} m & 2m \end{Bmatrix} \begin{Bmatrix} 1 \\ -\frac{1}{2} \end{Bmatrix} = m - m = 0$$

$$\{\phi_1\}^{\mathrm{T}}[k]\{\phi_2\} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 5k & -2k \\ -2k & 2k \end{bmatrix} \begin{Bmatrix} 1 \\ -\frac{1}{2} \end{Bmatrix} = \begin{Bmatrix} k & 2k \end{Bmatrix} \begin{Bmatrix} 1 \\ -\frac{1}{2} \end{Bmatrix} = k - k = 0$$

3.4 Concept of Normal Coordinates

It should be noted that the mode shapes can serve the same purpose as the trigonometric functions in Fourier series, and that any displacement vector $\{x\}$ for an N-degree-of-freedom structure can be developed by superposing suitable amplitudes of the N modes of vibration. Consider, for example, the three degree of freedom system shown in Fig.3.4.1. We have,

$$\{x\} = \begin{Bmatrix} x_{\mathrm{a}} \\ x_{\mathrm{b}} \\ x_{\mathrm{c}} \end{Bmatrix} = x_1^* \begin{Bmatrix} \phi_{\mathrm{a}1} \\ \phi_{\mathrm{b}1} \\ \phi_{\mathrm{c}1} \end{Bmatrix} + x_2^* \begin{Bmatrix} \phi_{\mathrm{a}2} \\ \phi_{\mathrm{b}2} \\ \phi_{\mathrm{c}2} \end{Bmatrix} + x_3^* \begin{Bmatrix} \phi_{\mathrm{a}3} \\ \phi_{\mathrm{b}3} \\ \phi_{\mathrm{c}3} \end{Bmatrix} = \begin{bmatrix} \phi_{\mathrm{a}1} & \phi_{\mathrm{a}2} & \phi_{\mathrm{a}3} \\ \phi_{\mathrm{b}1} & \phi_{\mathrm{b}2} & \phi_{\mathrm{b}3} \\ \phi_{\mathrm{c}1} & \phi_{\mathrm{c}2} & \phi_{\mathrm{c}3} \end{bmatrix} \begin{Bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{Bmatrix} \tag{3.4.1a}$$

or

$$\{x\} = [\phi]\{x^*\} \tag{3.4.1b}$$

where $\{x\}$, $[\phi]$ and $\{x^*\}$ are the displacement vector, mode shape matrix and normal coordinate vector, respectively.

Then, consider the undamped equation of motion of the system.

$$[m]\{\ddot{x}\} + [k]\{x\} = \{p\} \tag{3.4.2}$$

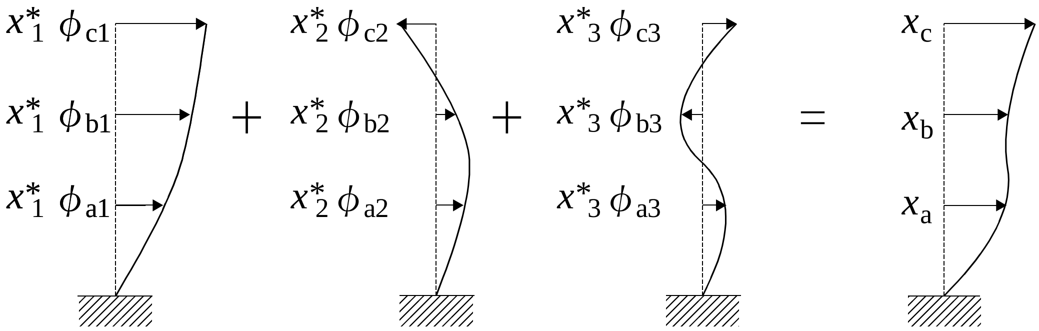


Fig.3.4.1 Concept of normal coordinates

Introducing Eq.(3.4.1b) and its second time derivative $\{\ddot{x}\} = [\phi]\{\ddot{x}^*\}$ leads to

$$[m][\phi]\{\ddot{x}^*\} + [k][\phi]\{x^*\} = \{p\} \tag{3.4.3}$$

Pre-multiplying both sides of this equation by the transpose of the j-th mode shape vector $\{\phi_j\}^{\mathrm{T}}$ yields

$$\{\phi_j\}^{\mathrm{T}}[m][\phi]\{\ddot{x}^*\} + \{\phi_j\}^{\mathrm{T}}[k][\phi]\{x^*\} = \{\phi_j\}^{\mathrm{T}}\{p\} \tag{3.4.4}$$

Expanding the first term of the left hand side, we have

$$\underbrace{\{\phi_j\}^{\mathrm{T}}[m]\{\phi_1\}\ddot{x}_1^* + \{\phi_j\}^{\mathrm{T}}[m]\{\phi_2\}\ddot{x}_2^* + \ldots}_{0} + \{\phi_j\}^{\mathrm{T}}[m]\{\phi_j\}\ddot{x}_j^*$$

$$+ \underbrace{\ldots + \{\phi_j\}^{\mathrm{T}}[m]\{\phi_n\}\ddot{x}_n^*}_{0} = \{\phi_j\}^{\mathrm{T}}[m]\{\phi_j\}\ddot{x}_j^* \tag{3.4.5}$$

Remembering the orthogonality conditions in the previous section, the above equation becomes as follows.

$$\{\phi_j\}^{\mathrm{T}}[m]\{\phi_j\}\ddot{x}_j^* + \{\phi_j\}^{\mathrm{T}}[k]\{\phi_j\}x_j^* = \{\phi_j\}^{\mathrm{T}}\{p\} \tag{3.4.6}$$

If we define new symbols as

$$m_j^* = \{\phi_j\}^{\mathrm{T}}[m]\{\phi_j\} \tag{3.4.7a}$$

$$k_j^* = \{\phi_j\}^{\mathrm{T}}[k]\{\phi_j\} \tag{3.4.7b}$$

$$p_j^* = \{\phi_j\}^{\mathrm{T}}\{p\} \tag{3.4.7c}$$

which are called the (normal coordinate) generalized mass, generalized stiffness and generalized load for the j-th mode, respectively. Then, Eq.(3.4.6) can be written,

$$m_j^*\,\ddot{x}_j^* + k_j^*\,x_j^* = p_j^* \tag{3.4.8}$$

which is a SDOF equation of motion for the j-th mode. If the mode shape vector form of Eq.(3.3.1), $[k]\{\phi_j\} = \omega_j^2[m]\{\phi_j\}$, is pre-multiplied on both sides by $\{\phi_j\}^{\mathrm{T}}$, the generalized stiffness for the j-th mode is related to the generalized mass by the frequency of vibration as follows.

$$\{\phi_j\}^{\mathrm{T}}[k]\{\phi_j\} = \omega_j^2\{\phi_j\}^{\mathrm{T}}[m]\{\phi_j\} \tag{3.4.9a}$$

or

$$k_j^* = \omega_j^2\, m_j^* \tag{3.4.9b}$$

Then Eq.(3.4.8) can be rewritten

$$\ddot{x}_j^* + \omega_j^2\, x_j^* = \frac{p_j^*}{m_j^*} \tag{3.4.10}$$

The equation of motion of the damped system is

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = \{p\} \tag{3.4.11}$$

If it is assumed that the damping matrix satisfies the following orthogonality condition,

$$\{\phi_j\}^{\mathrm{T}}[c]\{\phi_k\} = 0 \qquad\qquad j \neq k \tag{3.4.12}$$

we obtain a SDOF equation of the motion for the j-th mode as follows.

$$m_j^*\ddot{x}_n^* + c_j^*\dot{x}_j^* + k_j^*x_j^* = p_j^* \tag{3.4.13}$$

where

$$c_j^* = \{\phi_j\}^{\mathrm{T}}[c]\{\phi_j\} \tag{3.4.14}$$

is the generalized damping for the j-th mode. Then Eq.(3.4.13) can be rewritten as

$$\ddot{x}_j^* + 2\omega_j\zeta_j\dot{x}_j^* + \omega_j^2x_j^* = \frac{p_n^*}{m_j^*} \tag{3.4.15}$$

where

$$\zeta_j = \frac{c_j^*}{2m_j^*\omega_j}$$

is the j-th mode damping ratio.

The procedure described above can be used to obtain an independent SDOF equation for each mode of vibration. That is, the normal coordinates transform a set of n simultaneous differential equations into a set of independent normal coordinate equations. Therefore, the dynamic response can be obtained by solving separately for the response of each normal coordinate and then superposing them. This procedure is called the mode superposition method, and will be summarized in Section 3.8.

3.5 Damping Models

A dashpot is commonly used to represent the mechanism of structural damping. The dashpot has been tacitly combined with the spring in parallel. However, a number of combinations of dashpot and spring for dynamic systems have been proposed. We will take a brief look at several simple models of the dashpot and spring.

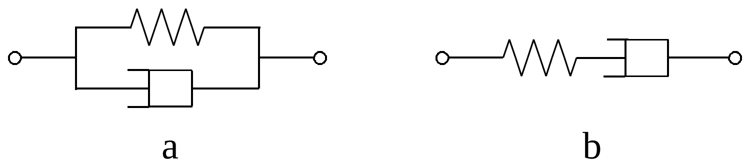


Fig.3.5.1 (a) Damping for Voigt model and (b) Maxwell model

(1) Voigt Model

The combination of dashpot and spring in parallel as shown in Fig.3.5.1a is called Voigt model, which is sometimes referred to as Kelvin model. Let the spring constant, damping coefficient, force applied to the system, and resulting deformation be denoted by k, c, p, and δ, respectively. Since equal deformations arise in the spring and the dashpot for a parallel model, the force of the spring is $p_1 = k\,\delta$, and the force of dashpot is $p_2 = c\,\dot{\delta}$. Then, from $p = p_1 + p_2$, we have

$$p = k\,\delta + c\,\dot{\delta} \tag{3.5.1}$$

Assuming that a harmonic deformation with frequency ω is given to the model,

$$\delta = \delta_0 \sin\omega t \tag{3.5.2}$$

Then,

$$p = \delta_0 (k \sin\omega t + c\,\omega \cos\omega t) \tag{3.5.3}$$

Eliminating ωt from the above two equations, the relationship between force p and deformation δ is given as follow.

$$\left(\frac{p}{k\,\delta_0}\right)^2 - 2\left(\frac{p}{k\,\delta_0}\right)\left(\frac{\delta}{\delta_0}\right) + \left\{1 + \left(\frac{c\,\omega}{k}\right)^2\right\}\left(\frac{\delta}{\delta_0}\right)^2 = \left(\frac{c\,\omega}{k}\right)^2 \tag{3.5.4a}$$

This represents the equation of an ellipse (Fig.3.5.2) that is shown as follows.

Let us denote $Y = \dfrac{p}{k\,\delta_0}$, $X = \dfrac{\delta}{\delta_0}$ and $d = \dfrac{c\,\omega}{k}$, then Eq.(3.5.4a) becomes

$$Y^2 - 2\,YX + (1 + d^2)X^2 = d^2 \tag{3.5.4b}$$

The X-Y axes is rotated by an angle θ to a new x-y axes. Then the relationships between these two pairs of coordinates are as follows (Fig.3.5.3).

$$X = x\cos\theta - y\sin\theta \tag{3.5.5a}$$

$$Y = x\sin\theta + y\cos\theta \tag{3.5.5b}$$

Substituting the above two equations into Eq.(3.5.4b), we have

$$(x\sin\theta + y\cos\theta)^2 - 2(x\sin\theta + y\cos\theta)(x\cos\theta - y\sin\theta) + (1 + d^2)(x\cos\theta - y\sin\theta)^2 = d^2$$

$$\begin{aligned} &x^2\sin^2\theta + 2xy\sin\theta\cos\theta + y^2\cos^2\theta \\ &-2x^2\sin\theta\cos\theta + 2xy\sin^2\theta - 2xy\cos^2\theta + 2y^2\sin\theta\cos\theta \\ &+(1 + d^2)(x^2\cos^2\theta - 2xy\sin\theta\cos\theta + y^2\sin^2\theta) = d^2 \end{aligned}$$

Using the formulae $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$, the above equation becomes

$$(1 - \sin 2\theta + d^2\frac{1+\cos 2\theta}{2})x^2 + (1 + \sin 2\theta + d^2\frac{1-\cos 2\theta}{2})y^2 - (2\cos 2\theta + d^2\sin 2\theta)xy = d^2 \tag{3.5.6}$$

In order to eliminate the term that includes xy, let us rotate the X-Y axes by an angle θ that satisfies the following equations.

$$\tan 2\theta = -\frac{2}{d^2} \qquad \text{or} \qquad \sin 2\theta = -\frac{2}{\sqrt{4+d^4}} \qquad \cos 2\theta = \frac{d^2}{\sqrt{4+d^4}} \tag{3.5.7}$$

Then, Eq.(3.5.6) becomes the equation that represents an ellipse as follows.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{3.5.8}$$

where,

$$a^2 = \frac{2d^2}{2 + d^2 - \sqrt{4+d^4}} \tag{3.5.9a}$$

$$b^2 = \frac{2d^2}{2 + d^2 + \sqrt{4+d^4}} \tag{3.5.9b}$$

The area enclosed by the ellipse is $\pi ab = \pi d$ for the coordinates of x and y. Then the area of the inclined ellipse in the original coordinates of p and δ (Fig.3.5.2) is,

$$\Delta E = \pi d(k\,\delta_0)\delta_0 = \pi c\,\omega\delta_0^2 \tag{3.5.10}$$

The area represents the energy loss or energy dissipation per cycle that is proportional to the frequency ω.

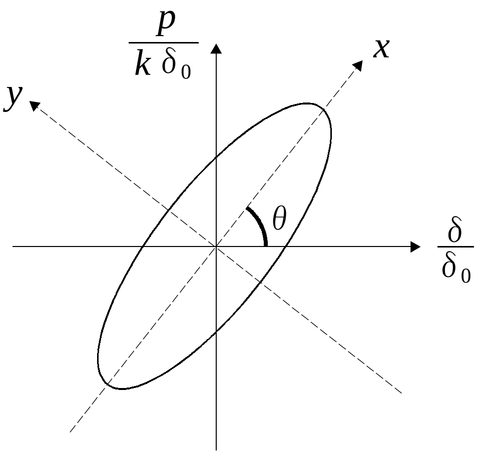


Fig.3.5.2 Ellipse representing force and deformation relationship

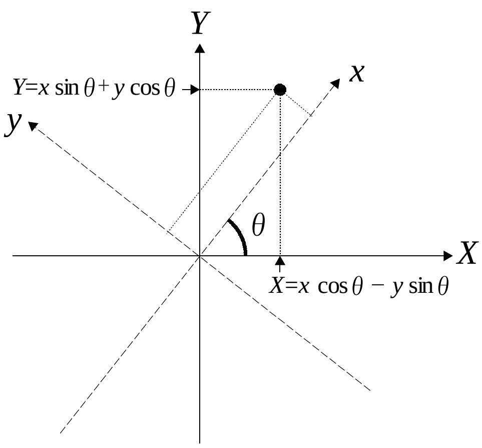


Fig.3.5.3 Rotation of axes

(2) Maxwell Model

Maxwell model is, on the other hand, the combination of spring and dashpot in series (Fig.3.5.1b). In the series model, the force of the spring is equal to the force of the dashpot, i.e. $p_1 = p_2 = p$. The deformation of the spring is

$$\delta_1 = \frac{p}{k} \tag{3.5.11}$$

The deformation of the dashpot is

$$\delta_2 = \int \left(\frac{p}{c}\right) dt \qquad \left(\text{since, } \dot{\delta}_2 = \frac{p}{c}\right) \tag{3.5.12}$$

From $\delta = \delta_1 + \delta_2$, the equation for Maxwell model is

$$\delta = \frac{p}{k} + \int \left(\frac{p}{c}\right) dt \tag{3.5.13}$$

Differentiating this equation yields

$$\dot{p} + \frac{k}{c} p = k\,\dot{\delta} \tag{3.5.14}$$

Assuming,

$$p = k\,\delta_0 \sin \omega t \tag{3.5.15}$$

we have

$$\delta = \delta_0 \Big(\sin \omega t - \frac{k}{c\,\omega} \cos \omega t \Big) \tag{3.5.16}$$

Eliminating ωt from the above two equations, we have

$$\left\{ 1 + \left(\frac{k}{c\,\omega}\right)^2 \right\} \left(\frac{p}{k\delta_0}\right)^2 - 2\left(\frac{p}{k\delta_0}\right)\left(\frac{\delta}{\delta_0}\right) + \left(\frac{\delta}{\delta_0}\right)^2 = \left(\frac{k}{c\,\omega}\right)^2 \tag{3.5.17a}$$

Let us again denote $Y = \dfrac{p}{k\,\delta_0}$, $X = \dfrac{\delta}{\delta_0}$ and $d = \dfrac{c\,\omega}{k}$, then Eq.(3.5.7a) becomes

$$\{1 + (1/d)^2\} Y^2 - 2\,YX + X^2 = (1/d)^2 \tag{3.5.17b}$$

Substituting Eqs.(3.5.5a) and (3.5.5b) to Eq.(3.5.17b), we have

$$\begin{aligned} &\Big\{ 1 - \sin 2\theta + (1/d)^2 \frac{1 - \cos 2\theta}{2} \Big\} x^2 \\ &+ \Big\{ 1 + \sin 2\theta + (1/d)^2 \frac{1 + \cos 2\theta}{2} \Big\} y^2 - \Big\{ 2 \cos 2\theta - (1/d)^2 \sin 2\theta \Big\} xy = (1/d)^2 \end{aligned} \tag{3.5.18}$$

In order to eliminate the term that includes xy, let the X-Y axes rotate by an angle θ that satisfies the following equations.

$$\tan 2\theta = \frac{2}{(1/d)^2} \qquad \text{or} \qquad \sin 2\theta = \frac{2}{\sqrt{4 + (1/d)^4}} \qquad \cos 2\theta = \frac{(1/d)^2}{\sqrt{4 + (1/d)^4}} \tag{3.5.19}$$

Then Eq.(3.5.18) becomes Eq.(3.5.8) that represents the equation of an ellipse, where

$$a^2 = \frac{2(1/d)^2}{2 + (1/d)^2 - \sqrt{4 + (1/d)^4}} \tag{3.5.20a}$$

$$b^2 = \frac{2(1/d)^2}{2 + (1/d)^2 + \sqrt{4 + (1/d)^4}} \tag{3.5.20b}$$

Therefore, the area of the ellipse in the coordinates of p and δ becomes as follows.

$$\Delta E = \pi(1/d)(k\delta_0)\delta_0 = \pi\frac{k^2\delta_0^2}{c\,\omega} \tag{3.5.21}$$

The area is inversely proportional to the frequency ω for Maxwell model, in contrast to the Voigt model.

(3) Hysteretic Damping Model

Although this model is apparently the same as Voigt model (Fig.3.5.1a), the damping coefficient c is assumed to be inversely proportional to the frequency ω. Therefore, the expansion of equations follows Voigt model with the substitution of $\hat{c}/\omega$ ($\hat{c}$: constant) in place of c for the damping coefficient.

Then, the energy loss for this case is

$$\Delta E = \pi\hat{c}\,{\delta_0}^2 \tag{3.5.22}$$

(4) Comparison of Damping Ratios

For each of the three models, the maximum potential energy is equal to

$$E_{\mathrm{P}} = \frac{1}{2}k\delta_0^2 \tag{3.5.23}$$

The energy loss rate (also called the specific energy loss), i.e. $\Delta E/E_{\mathrm{P}}$ is related to the damping factor,

$$\zeta = \frac{1}{4\pi}\frac{\Delta E}{E_{\mathrm{P}}} \tag{3.5.24}$$

Then, the damping ratio for each model can be obtained as follows.

Voigt model $$\zeta = \frac{1}{2}\frac{c\,\omega}{k} \tag{3.5.25a}$$

Maxwell model $$\zeta = \frac{1}{2}\frac{k}{c\,\omega} \tag{3.5.25b}$$

Hysteretic damping model $$\zeta = \frac{1}{2}\frac{\hat{c}}{k} \tag{3.5.25c}$$

These relationships are schematically shown in Fig.3.5.4. It should be noted that the damping factor ζ for Voigt model is proportional to the frequency ω, the damping factor for Maxwell model is inversely proportional to the frequency, and the damping factor for the hysteretic damping model is constant with respect to frequency.

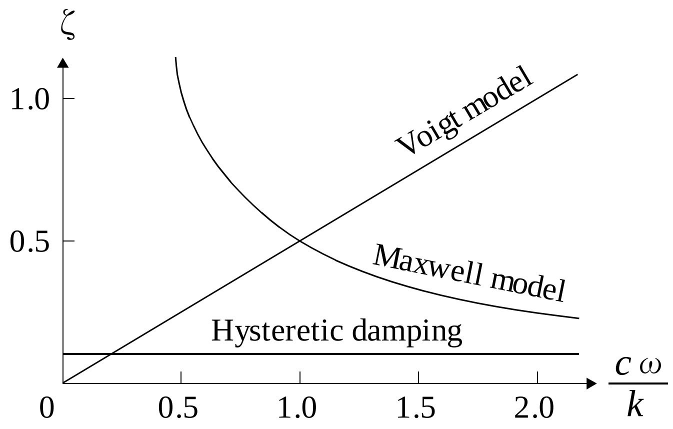


Fig.3.5.4 Frequency dependency of damping factor for each model

3.6 Stodola (Matrix Iteration) Method

This is one of the most convenient methods to solve for the mode shapes and corresponding natural frequencies of a structure. In Stodola method, the initially assumed mode shape is iteratively adjusted until an adequate approximation of the true mode shape has been achieved. Then the frequency of vibration is determined.

(1) Procedure of Stodola method

i) Dynamic matrix

The mode shape $\{u\}$ for an undamped system in free vibration can be written as

$$[k]\{u\} = \omega_j^2[m]\{u\} \tag{3.6.1}$$

Pre-multiplying the above equation by the flexibility matrix $[f] = [k]^{-1}$, we have

$$\begin{aligned}\{u\} &= \omega_j^2[f][m]\{u\}\\ &= \omega_j^2[A]\{u\}\end{aligned} \tag{3.6.2}$$

where $[A] = [f][m]$ is the dynamic matrix.

ii) Assumption of mode shape

Assume the mode shape $\{u\}^{(1)}$ that can be considered to be a close approximation to the first mode, where the amplitude is arbitrary.

iii) Assumption of next mode shape

Calculate the following formula.

$$\{\hat{u}_c\}^{(1)} = [A]\{u\}^{(1)} \tag{3.6.3}$$

And assume the next mode shape as follows.

$$\{u\}^{(2)} = \alpha_1\{\hat{u}\}^{(1)} \tag{3.6.4}$$

where α_1 is an arbitrary constant. If we choose α_1 to be $\{u_i\}^{(1)} = \{u_i\}^{(2)}$, α_1 eventually converges on $1/\omega_j^2$.

iv) Repetition of assumed mode shape

Repeat step iii) until the following relation is satisfied

$$\{\hat{u}\}^{(r)} \approx \alpha_r \{u\}^{(r-1)} \tag{3.6.5}$$

Then $\{\hat{u}\}^{(r)}$ is the mode shape vector and $\alpha_r = 1/\omega_j^2$ is the eigenvalue.

(2) Proof of Convergence

Any assumed mode shape can be expressed as

$$\{u\}^{(1)} = \{\phi_1\}x_1^* + \{\phi_2\}x_2^* + \{\phi_3\}x_3^* + \cdots \tag{3.6.6}$$

Eq.(3.6.3) becomes

$$\{\hat{u}\}^{(1)} = [A]\{u\} = [A]\{\phi_1\}x_1^* + [A]\{\phi_2\}x_2^* + [A]\{\phi_3\}x_3^* + \cdots$$

Remembering $[A]\{\phi_i\} = \frac{1}{\omega_i^2}\{\phi_i\}$, the above equation can be written as

$$\{\hat{u}\}^{(1)} = \frac{1}{\omega_1^2}\{\phi_1\}x_1^* + \frac{1}{\omega_2^2}\{\phi_2\}x_2^* + \frac{1}{\omega_3^2}\{\phi_3\}x_3^* + \cdots \tag{3.6.7}$$

Repeating step iii) in (1), we have

$$\begin{aligned}
\{\hat{u}\}^{(2)} &= \alpha_1\Big\{\frac{1}{\omega_1^2}[A]\{\phi_1\}x_1^* + \frac{1}{\omega_2^2}[A]\{\phi_2\}x_2^* + \frac{1}{\omega_3^2}[A]\{\phi_3\}x_3^* + \cdots\Big\} \\
&= \alpha_1\Big\{\big(\frac{1}{\omega_1^2}\big)^2\{\phi_1\}x_1^* + \big(\frac{1}{\omega_2^2}\big)^2\{\phi_2\}x_2^* + \big(\frac{1}{\omega_3^2}\big)^2\{\phi_3\}x_3^* + \cdots\Big\} && (3.6.8)\\
\{\hat{u}\}^{(r)} &= \frac{\alpha_1\alpha_2\cdots\alpha_{r-1}}{\omega_1^{2r}}\Big\{\{\phi_1\}x_1^* + \big(\frac{\omega_1}{\omega_2}\big)^{2r}\{\phi_2\}x_2^* + \big(\frac{\omega_1}{\omega_3}\big)^{2r}\{\phi_3\}x_3^* + \cdots\Big\} && (3.6.9)
\end{aligned}$$

Note the following relationship

$$1 >> \left(\frac{\omega_1}{\omega_2}\right)^{2r} >> \left(\frac{\omega_1}{\omega_3}\right)^{2r} >> \ldots \tag{3.6.10}$$

Therefore, we can eliminate the higher modes from the first assumption of the mode shape, and the contribution of the higher modes can be made as small as is desired by repeating step iii) in (1).

(3) Analysis of Higher Modes

The proof of convergence in (2) indicates that the procedure can be used to evaluate the higher modes as well. Eq.(3.6.9) shows that, if x_1^* is equal to zero, the process must converge towards the second mode shape. Thus, in order to calculate the second mode by the matrix iteration method, it is necessary to assume a trial mode shape $\{u_2\}^{(1)}$ which contains no first mode component as follows.

Any arbitrary assumption of the second mode shape is

$$\{u_2\}^{(1)} = \{\phi_1\}x_1^* + \{\phi_2\}x_2^* + \{\phi_3\}x_3^* + \dots \tag{3.6.11}$$

Pre-multiplying both sides by $\{\phi_1\}^{\mathrm{T}}[m]$ leads to

$$\begin{aligned}\{\phi_1\}^{\mathrm{T}}[m]\{u_2\}^{(1)} &= \{\phi_1\}^{\mathrm{T}}[m]\{\phi_1\}x_1^* + \{\phi_1\}^{\mathrm{T}}[m]\{\phi_2\}x_2^* + \{\phi_1\}^{\mathrm{T}}[m]\{\phi_3\}x_3^* + \cdots \\ &= \{\phi_1\}^{\mathrm{T}}[m]\{\phi_1\}x_1^* \end{aligned}\tag{3.6.12}$$

Hence,

$$x_1^* = \frac{\{\phi_1\}^{\mathrm{T}}[m]\{u_2\}^{(1)}}{\{\phi_1\}^{\mathrm{T}}[m]\{\phi_1\}} \tag{3.6.13}$$

Therefore, if this component is removed from the assumed shape, the vector is to be purified.

$$\{\tilde{u}_2\}^{(1)} = \{u_2\}^{(1)} - \{\phi_1\}x_1^* \tag{3.6.14}$$

This purified vector will now converge towards the second mode shape in the matrix iteration method. Round-off error in the numerical calculations, however, will cause reappearance of the first mode. Therefore this purification operation is necessary at each cycle of iteration. A convenient means of purification is to use a sweeping matrix. Substituting Eq.(3.6.13) to Eq.(3.6.14) yields

$$\begin{aligned}\{\tilde{u}_2\}^{(1)} &= \{u_2\}^{(1)} - \frac{\{\phi_1\}\{\phi_1\}^{\mathrm{T}}[m]\{u_2\}^{(1)}}{\{\phi_1\}^{\mathrm{T}}[m]\{\phi_1\}} \\ &= [s_1]\{u_2\}^{(1)}\end{aligned} \tag{3.6.15}$$

where

$$[s_1] = [I] - \frac{\{\phi_1\}\{\phi_1\}^{\mathrm{T}}[m]}{\{\phi_1\}^{\mathrm{T}}[m]\{\phi_1\}} \tag{3.6.16}$$

is the sweeping matrix.

Then, the Stodola procedure in (1) can now be formulated so that it converges towards the second mode, replacing the dynamic matrix [A] by the product [A][S] of the dynamic matrix and the sweeping matrix.

The third and higher modes can be obtained by the use of the following sweeping matrices.

$$[s_2] = [s_1] - \frac{\{\phi_2\}\{\phi_2\}^{\mathrm{T}}[m]}{\{\phi_2\}^{\mathrm{T}}[m]\{\phi_2\}} \tag{3.6.17}$$

$$[s_3] = [s_2] - \frac{\{\phi_3\}\{\phi_3\}^{\mathrm{T}}[m]}{\{\phi_3\}^{\mathrm{T}}[m]\{\phi_3\}} \tag{3.6.18}$$

[Example 3.4]

For a dynamic matrix $[A]$ given below, calculate the mode shape vector and the eigen value of the first mode by Stodola Method.

$$[A] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

[Solution]

Table E3.4 Calculation sheet for Stodola method

$[A]$			$\{\phi\}$					
			(1)	(2)	(3)	(4)	(5)	(6)
1	1	1	1	0.500	0.452	0.446	0.445	0.445
1	2	2	1	0.833	0.806	0.803	0.802	0.802
1	2	3	1	1	1	1	1	1
$1/\omega^2$			6	5.166	5.064	5.052	5.049	5.049

Therefore $\{\phi\}^T = \{0.445\ 0.802\ 1\}$ and $1/\omega^2 = 5.049$.
Then, $\omega^2 = 0.198$, $\omega = 0.445$ and $T = 14.12$.

[Example 3.5]

For a five story shear type structure, calculate the mode shape vector and the eigen value of the first mode by Stodola Method. Where, $W_1 = W_2 = W_3 = W_4 = W_5 = 490$ (kN), $k_1 = 500$ (kN/cm), $k_2 = 450$ (kN/cm), $k_3 = 400$ (kN/cm), $k_4 = 350$ (kN/cm), $k_5 = 300$ (kN/cm),

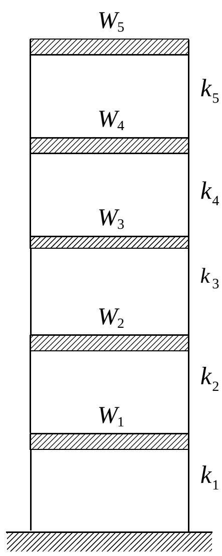


Fig.E3.5

[Solution]

Table E3.5 Calculation sheet for Stodola method

r	i	ϕ_{1i}	m_i	$m_i\phi_{1i}$	$\sum_n^i m_i\phi_{1i}$	k_i	δ_i *	$\sum_n^i \delta_i$	ω_1^2 **	ϕ_{1i}
	5	5	0.5	2.500	2.500	300	0.00833	0.06675	74.91	4.450
	4	4	0.5	2.000	4.500	350	0.01286	0.05842	68.47	3.895
1	3	3	0.5	1.500	6.000	400	0.01500	0.04556	65.85	3.037
	2	2	0.5	1.000	7.000	450	0.01556	0.03056	65.45	2.037
	1	1	0.5	0.500	7.500	500	0.01500	0.01500	66.67	1.000
	5	4.450		2.225	2.225		0.00742	0.06290	70.74	4.362
	4	3.895		1.948	4.173		0.01192	0.05548	70.21	3.847
2	3	3.037	ditto	1.519	5.692	ditto	0.01423	0.04356	69.72	3.021
	2	2.037		1.019	6.711		0.01491	0.02933	69.45	2.034
	1	1.000		0.500	7.211		0.01442	0.01442	69.35	1.000
	5	4.362		2.181	2.181		0.00727	0.06205	70.30	4.348
	4	3.847		1.924	4.105		0.01173	0.05478	70.23	3.839
3	3	3.021	ditto	1.511	5.616	ditto	0.01404	0.04305	70.17	3.017
	2	2.034		1.017	6.633		0.01474	0.02901	70.11	2.033
	1	1.000		0.500	7.133		0.01427	0.01427	70.08	1.000
	5	4.348		2.174	2.174		0.00725	0.06191	70.23	4.348
	4	3.839		1.920	4.094		0.01170	0.05466	70.23	3.838
4	3	3.017	ditto	1.509	5.603	ditto	0.01401	0.04296	70.23	3.017
	2	2.033		1.017	6.620		0.01471	0.02895	70.22	2.033
	1	1.000		0.500	7.120		0.01424	0.01424	70.22	1.000
	5	4.348		2.174	2.174		0.00725	0.06190	70.24	4.347
	4	3.838		1.919	4.093		0.01169	0.05465	70.23	3.838
5	3	3.017	ditto	1.509	5.602	ditto	0.01401	0.04296	70.23	3.017
	2	2.033		1.017	6.619		0.01471	0.02895	70.22	2.033
	1	1.000		0.500	7.119		0.01424	0.01424	70.22	1.000
	5	4.347		2.174	2.174		0.00725	0.06190	70.23	4.347
	4	3.838		1.919	4.093		0.01169	0.05465	70.23	3.838
6	3	3.017	ditto	1.509	5.602	ditto	0.01401	0.04296	70.23	3.017
	2	2.033		1.017	6.619		0.01471	0.02895	70.22	2.033
	1	1.000		0.500	7.119		0.01424	0.01424	70.22	1.000

* $\delta_i = \sum_n^i m_i\phi_{1i}/k_i$, ** $\omega_1^2 = \phi_i/(\sum_n^i \delta_i)$

Therefore, $\{\phi\}^{\mathrm{T}} = \{1 \quad 2.03 \quad 3.02 \quad 3.84 \quad 4.35\}^{\mathrm{T}}$, $\omega^2 = 70.22$, $\omega = 8.38$(rad/s), and $T = 0.75$(s).

3.7 Holzer Method

In Stodola method, an assumed mode shape is adjusted until the true mode shape is obtained, and then the frequency of vibration is calculated. In the Holzer method, the process is essentially the reverse. The frequency is adjusted so that the true frequency is established and the mode shape is evaluated simultaneously. It should be noted that this method is only applicable to so-called shear type structures or chain structures. The process is as follows. (See Fig.3.7.1)

i) First of all, assume ω_j, where it can be an approximation of frequency for any mode.
ii) The force p_n produced by the n-th mass is equal to the shear force V_n just below the n-th mass, and is given by

$$p_n = V_n = \omega_j^2 \, m_n \, u_n$$

where u_n can be an arbitrary constant.
iii) The deflection δ_n caused by the shear force V_n is

$$\delta_n = V_n / k_n = \omega_j^2 m_n \, u_n / k_n = u_n - u_{n-1}$$

iv) Then, the $(n-1)$-th element of the mode shape vector is

$$u_{n-1} = u_n - \delta_n$$

v) The shear force V_{n-1} just below the $(n-1)$-th mass is

$$V_{n-1} = p_n + p_{n-1} = \sum_{i=n-1}^{n} \omega_j^2 m_i \, u_i$$

vi) Then, the deflection δ_{n-1} caused by the shear force V_{n-1} is

$$\delta_{n-1} = V_{n-1} / k_{n-1} = u_{n-1} - u_{n-2}$$

vii) The $(n-2)$-th element of the mode shape vector is

$$u_{n-2} = u_{n-1} - \delta_{n-1}$$

Repeating steps v)$\sim$vii), we have
viii)

$$\delta_1 = \frac{V_1}{k_1} = \sum_{i=1}^{n} \omega_j^2 m_i \, u_i / k_i$$

ix)

$$u_0 = u_1 - \delta_1$$

If u_0 is equal to zero, the initial assumption of ω_j is the correct frequency of vibration. If this value is positive, the assumed ω_j is smaller than the true ω_n in the case of odd modes, or the assumed ω_j is larger than the true ω_j in case of even modes.

The Holzer method explained above can be conveniently performed according to Table 3.7.1.

The procedure for filling the blanks of the table is as follows.
a) Write in each value of mass m_i and stiffness k_i.
b) Assuming $u_n = 1$, calculate p_n, V_n and δ_n.
c) Calculate u_{n-1} where $u_{n-1} = u_n - \delta_n$.
d) Repeat steps b) and c) until $u_0 = u_1 - \delta_1$ is calculated.

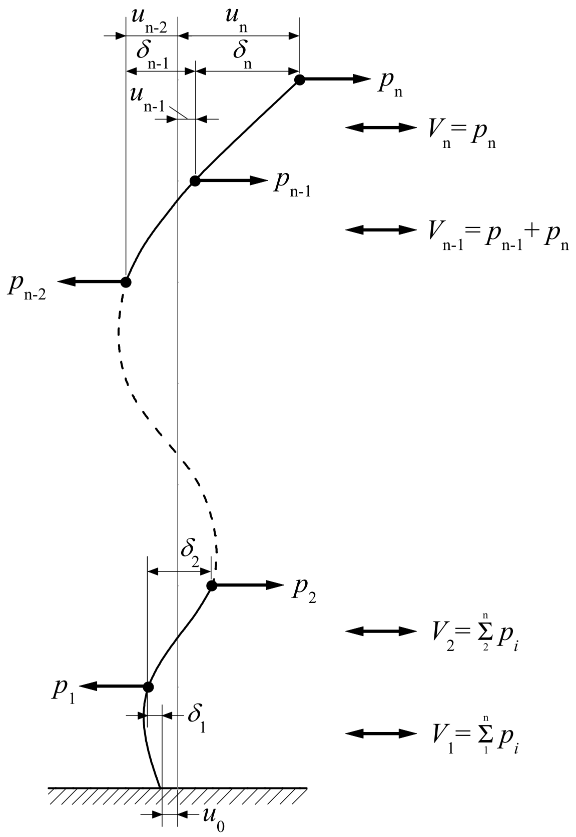


Fig.3.7.1 Holzer method

Table 3.7.1 Calculation sheet for Holzer method

i	m_i	u_i		$p_i = m_i u_i \omega_j^2$		$V_i = \sum p_i$		k_i		$\delta_i = V_i/k_i$
n		1.000	$\rightarrow$		$\rightarrow$		$\rightarrow$		$\rightarrow$	
$n-1$										
$n-2$										
$\vdots$										
3										
2										
1										

[Example 3.6]

For a five story shear type structure as shown in Fig.E3.6, calculate the mode shape vector and the eigen value of the first mode and the second mode by Holzer method, where, $W_1 = W_2 = W_3 = W_4 = W_5 = 490$ (kN), $k_1 = 500$ (kN/cm), $k_2 = 450$ (kN/cm), $k_3 = 400$ (kN/cm), $k_4 = 350$ (kN/cm), $k_5 = 300$ (kN/cm).

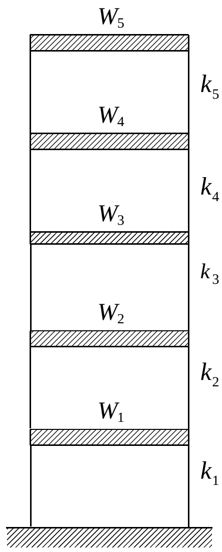


Fig.E3.6

[Solution]

For SDOF systems, the natural period is given as

$$T_\mathrm{n} = 2\pi\sqrt{\frac{m}{k}}$$

Let the displacement of the system be δ_{st}, when the weight of the system is applied horizontally.

$$k = \frac{mg}{\delta_{st}}$$

Therefore,

$$T_\mathrm{n} = 2\pi\sqrt{\frac{\delta_{st}}{g}} \approx \frac{\sqrt{\delta_{st}}}{5}$$

where, T_n is given in second and δ_{st} is given in cm.

For MDOF systems, similar formula is given as follows.

$$T_1 \approx \frac{\sqrt{\delta_{st}}}{5.5}$$

The horizontal displacement of the structure subjected to horizontal forces that are equal to its own weight is

$$\begin{aligned}\delta_{st} &= \frac{490}{300} + \frac{490 \times 2}{350} + \frac{490 \times 3}{400} + \frac{490 \times 4}{450} + \frac{490 \times 5}{500} \\ &= 1.63 + 2.80 + 3.68 + 4.36 + 4.90 = 17.37(\mathrm{cm})\end{aligned}$$

$$T_1 \approx \frac{\sqrt{17.37}}{5.5} \approx 0.76(\mathrm{s}) \quad \omega \approx 8.27 \quad \omega^2 \approx 68$$

Table E3.6a First mode for Holzer method
(First assumption, $\omega_1^2 = 68$)

i	m_i	u_i	p_i $= m_i u_i \omega_j^2$	V_i $= \sum p_i$	k_i	δ_i $= V_i/k_i$
5	0.500	1.000	34.00	34.00	300	0.113
4	0.500	0.887	30.16	64.16	350	0.183
3	0.500	0.704	23.94	88.10	400	0.220
2	0.500	0.484	16.46	104.56	450	0.232
1	0.500	0.252	8.57	113.13	500	0.226
	$u_0 =$	0.026				

Table E3.6b First mode for Holzer method
(Second assumption, $\omega_1^2 = 70$)

i	m_i	u_i	p_i $= m_i u_i \omega_j^2$	V_i $= \sum p_i$	k_i	δ_i $= V_i/k_i$
5	0.500	1.000	35.00	35.00	300	0.117
4	0.500	0.883	30.91	65.91	350	0.188
3	0.500	0.695	24.33	90.24	400	0.226
2	0.500	0.469	16.42	106.66	450	0.237
1	0.500	0.232	8.12	114.78	500	0.230
	$u_0 =$	0.002				

Table E3.6c First mode for Holzer method
(Third assumption, $\omega_1^2 = 70.2$)

i	m_i	u_i	p_i $= m_i u_i \omega_j^2$	V_i $= \sum p_i$	k_i	δ_i $= V_i/k_i$
5	0.500	1.000	35.10	35.10	300	0.117
4	0.500	0.883	30.99	66.09	350	0.189
3	0.500	0.694	24.36	90.45	400	0.226
2	0.500	0.468	16.43	106.88	450	0.238
1	0.500	0.230	8.07	114.95	500	0.230
	$u_0 =$	0.000				

Therefore, for the first mode
$\omega_1^2 = 70.2$, $\omega_1 = 8.38(\text{rad/s})$, $T_1 = 0.75(\text{s})$
$\{u_1\}^{\mathrm{T}} = \{0.230 \;\; 0.468 \;\; 0.694 \;\; 0.883 \;\; 1.000\}^{\mathrm{T}}$

In general, the natural frequencies of a uniform continuous beam fixed at the end is given as follows.

$$\omega_j = (2j-1)\omega_1$$

Therefore, it may be assumed that the natural frequency of the second mode is $\omega_2 \approx 3\omega_1 \approx 3 \times 8.38 \approx 25.1$. Then, let us assume $\omega_2^2 \approx 25.1^2 \approx 630 \approx 600$.

Table E3.6d Second mode for Holzer method
(First assumption, $\omega_2^2 = 600$)

i	m_i	u_i	p_i $= m_i u_i \omega_j^2$	V_i $= \sum p_i$	k_i	δ_i $= V_i/k_i$
5	0.500	1.000	300.00	300.00	300	1.000
4	0.500	0.000	0.00	300.00	350	0.857
3	0.500	−0.857	−257.14	42.86	400	0.107
2	0.500	−0.964	−289.24	−246.38	450	−0.548
1	0.500	−0.416	−124.95	−371.33	500	−0.743

$u_0 = 0.327$

Table E3.6e Second mode for Holzer method
(Second assumption, $\omega_2^2 = 500$)

i	m_i	u_i	p_i $= m_i u_i \omega_j^2$	V_i $= \sum p_i$	k_i	δ_i $= V_i/k_i$
5	0.500	1.000	250.00	250.00	300	0.833
4	0.500	0.167	41.67	291.67	350	0.833
3	0.500	−0.667	−166.67	125.00	400	0.313
2	0.500	−0.979	−244.79	−119.79	450	−0.266
1	0.500	−0.713	−178.79	−298.03	500	−0.596

$u_0 = -0.117$

Table E3.6f Second mode for Holzer method
(Third assumption, $\omega_2^2 = 500 + \frac{100 \times 0.117}{0.117 + 0.327} = 526$)

i	m_i	u_i	p_i $= m_i u_i \omega_j^2$	V_i $= \sum p_i$	k_i	δ_i $= V_i/k_i$
5	0.500	1.000	263.00	263.00	300	0.877
4	0.500	0.123	32.35	295.35	350	0.844
3	0.500	−0.721	−189.62	105.73	400	0.264
2	0.500	−0.985	−259.06	−153.33	450	−0.341
1	0.500	−0.644	−169.37	−323.70	500	−0.645

$u_0 = 0.001$

Therefore, for the second mode
$\omega_2^2 = 526$, $\omega_2 = 22.93$(rad/s), $T_2 = 0.274$(s),
$\{u_2\}^{\mathrm{T}} = \{-0.644 \ \ -0.985 \ \ -0.721 \ \ 0.123 \ \ 1.000\}^{\mathrm{T}}$.

3.8 Mode Superposition and Modal Analysis

(1) Derivation of Equations

The procedures for mode superposition and modal analysis are explained, deriving equations for a lumped mass "n" degrees of freedom as an example.

The equation of motion for a MDOF system is as already given

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = \{p\} \tag{3.8.1a}$$

When the system is subjected to earthquake ground acceleration $\ddot{x}_{\mathrm{g}}(t)$ at the base, the above equation is expressed as follows.

$$[m]\{\ddot{x}+\ddot{x}_{\mathrm{g}}\}+[c]\{\dot{x}\}+[k]\{x\}=\{0\} \tag{3.8.1b}$$

The base acceleration term is moved to the right hand side, and then

$$[m]\{\ddot{x}\}+[c]\{\dot{x}\}+[k]\{x\}=-[m]\{1\}\ddot{x}_{\mathrm{g}} \tag{3.8.1c}$$

where $\{1\}$ is a vector of ones, i.e. all elements of the vector are unity.

The matrices for a lumped mass n degrees of freedom are given as follows.

$$[m]=\begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix}=\begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & m_3 \end{bmatrix} \tag{3.8.2a}$$

$$[k]=\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix} \tag{3.8.2b}$$

$$[c]=\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} \tag{3.8.2c}$$

The equation of motion of undamped free vibration is

$$[m]\{\ddot{x}\}+[k]\{x\}=\{0\} \tag{3.8.3}$$

As was used for the SDOF system, we assume that the displacement vector is expressed as a product of constant vector and time function $e^{i\omega t}$ *, i.e.

$$\{x\}=\{u\}e^{i\omega t} \tag{3.8.4a}$$

namely,

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}=\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}e^{i\omega t} \tag{3.8.4b}$$

In order to obtain a non-zero vector solution for $\{u\}$, the determinant of the expression $\{[k]-\omega^2[m]\}$ must be zero, i.e.

$$\Big|[k]-\omega^2[m]\Big|=0 \tag{3.8.5}$$

* $e^{\pm i\omega}=\cos\omega\pm i\sin\omega$

From this condition, we obtain n values of ω_j^2 as eigenvalues. In addition, a corresponding vector can be computed for each eigenvalue.

Dividing each element of the vector $\{u\}$ by one reference element (usually the first or the largest), dimensionless vector $\{\phi_j\}$ is obtained. Combining the eigen vectors, a mode shape matrix is obtained as follows.

$$[\phi] = [\{\phi_1\}\{\phi_2\}\cdots\{\phi_n\}] = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{bmatrix} \tag{3.8.6}$$

Now, we assume that the solution $\{x\}$ for the original problem is expressed as a product of $[\phi]$ and a time function vector $\{x^*\}$.

$$\{x\} = [\phi]\{x^*\} = [\{\phi_1\}\ \{\phi_2\}\ \cdots\ \{\phi_n\}] \begin{Bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{Bmatrix} \tag{3.8.7}$$

Then the equation of motion becomes

$$[m][\phi]\{\ddot{x}^*\} + [c][\phi]\{\dot{x}^*\} + [k][\phi]\{x^*\} = -[m]\{1\}\ddot{x}_{\mathrm{g}} \tag{3.8.8}$$

Pre-multiplying the transposed matrix of $[\phi]$ to each term of the above equation, we have

$$[\phi]^{\mathrm{T}}[m][\phi]\{\ddot{x}^*\} + [\phi]^{\mathrm{T}}[c][\phi]\{\dot{x}^*\} + [\phi]^{\mathrm{T}}[k][\phi]\{x^*\} = -[\phi]^{\mathrm{T}}[m]\{1\}\ddot{x}_{\mathrm{g}} \tag{3.8.9}$$

As it was mentioned previously, i.e. because of orthogonality of eigen vectors, we have

$$[\phi]^{\mathrm{T}}[m][\phi] = \begin{bmatrix} m_1^* & 0 & \cdots & 0 \\ 0 & m_2^* & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & m_n^* \end{bmatrix} \tag{3.8.10}$$

where, $m_j^* = \{\phi_j\}^{\mathrm{T}}[m]\{\phi_j\}$ is called the generalized mass.

$$[\phi]^{\mathrm{T}}[k][\phi] = \begin{bmatrix} k_1^* & 0 & \cdots & 0 \\ 0 & k_2^* & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & k_n^* \end{bmatrix} \tag{3.8.11}$$

where, $k_j^* = \{\phi_j\}^{\mathrm{T}}[k]\{\phi_j\}$ is called the generalized stiffness.

Generally the damping matrix is not diagonalized by matrix $[\phi]$, since eigen vectors are not orthogonal with respect to damping matrix. Therefore,

$$[\phi]^{\mathrm{T}}[c][\phi] = \begin{bmatrix} c_{11}^* & c_{12}^* & \cdots & c_{1n}^* \\ c_{21}^* & c_{22}^* & \cdots & c_{2n}^* \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1}^* & c_{n2}^* & \cdots & c_{nn}^* \end{bmatrix} \tag{3.8.12a}$$

However, we assume that the above matrix becomes diagonal, i.e.,

$$[\phi]^{\mathrm{T}}[c][\phi] = \begin{bmatrix} c_1^* & 0 & \cdots & 0 \\ 0 & c_2^* & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & c_n^* \end{bmatrix} \tag{3.8.12b}$$

where, $c_j^* = \{\phi_j\}^{\mathrm{T}}[c]\{\phi_j\}$ is called the generalized damping.

Then, the equation of motion becomes

$$\begin{bmatrix} m_1^* & 0 & \cdots & 0 \\ 0 & m_2^* & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & m_n^* \end{bmatrix} \begin{Bmatrix} \ddot{x}_1^* \\ \ddot{x}_2^* \\ \vdots \\ \ddot{x}_n^* \end{Bmatrix} + \begin{bmatrix} c_1^* & 0 & \cdots & 0 \\ 0 & c_2^* & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & c_n^* \end{bmatrix} \begin{Bmatrix} \dot{x}_1^* \\ \dot{x}_2^* \\ \vdots \\ \dot{x}_n^* \end{Bmatrix} + \begin{bmatrix} k_1^* & 0 & \cdots & 0 \\ 0 & k_2^* & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & k_n^* \end{bmatrix} \begin{Bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{Bmatrix} = - \begin{Bmatrix} \sum_{i=1}^n m_i\phi_{i1} \\ \sum_{i=1}^n m_i\phi_{i2} \\ \vdots \\ \sum_{i=1}^n m_i\phi_{in} \end{Bmatrix} \ddot{x}_{\mathrm{g}} \tag{3.8.13a}$$

This matrix equation is equivalent to one of the following three sets of differential equations.

$$\begin{aligned} m_1^*\ddot{x}_1^* + c_1^*\dot{x}_1^* + k_1^*x_1^* &= -\ddot{x}_{\mathrm{g}} \textstyle\sum_{i=1}^n m_i\phi_{i1} \\ m_2^*\ddot{x}_2^* + c_2^*\dot{x}_2^* + k_2^*x_2^* &= -\ddot{x}_{\mathrm{g}} \textstyle\sum_{i=1}^n m_i\phi_{i2} \\ &\vdots \\ m_n^*\ddot{x}_n^* + c_n^*\dot{x}_n^* + k_n^*x_n^* &= -\ddot{x}_{\mathrm{g}} \textstyle\sum_{i=1}^n m_i\phi_{in} \end{aligned} \tag{3.8.13b}$$

or

$$\begin{aligned} \ddot{x}_1^* + \frac{c_1^*}{m_1^*}\dot{x}_1^* + \frac{k_1^*}{m_1^*}x_1^* &= -\ddot{x}_{\mathrm{g}}\frac{\sum_{i=1}^n m_i\phi_{i1}}{m_1^*} \\ \ddot{x}_2^* + \frac{c_2^*}{m_2^*}\dot{x}_2^* + \frac{k_2^*}{m_2^*}x_2^* &= -\ddot{x}_{\mathrm{g}}\frac{\sum_{i=1}^n m_i\phi_{i2}}{m_2^*} \\ &\vdots \\ \ddot{x}_n^* + \frac{c_n^*}{m_n^*}\dot{x}_n^* + \frac{k_n^*}{m_n^*}x_n^* &= -\ddot{x}_{\mathrm{g}}\frac{\sum_{i=1}^n m_i\phi_{in}}{m_n^*} \end{aligned} \tag{3.8.13c}$$

Therefore,

$$\begin{aligned} \ddot{x}_1^* + 2\zeta_1\omega_1\dot{x}_1^* + {\omega_1}^2 x_1^* &= -\beta_1\ddot{x}_{\mathrm{g}} \\ \ddot{x}_2^* + 2\zeta_2\omega_2\dot{x}_2^* + {\omega_2}^2 x_2^* &= -\beta_2\ddot{x}_{\mathrm{g}} \\ &\vdots \\ \ddot{x}_n^* + 2\zeta_n\omega_n\dot{x}_n^* + {\omega_n}^2 x_n^* &= -\beta_n\ddot{x}_{\mathrm{g}} \end{aligned} \tag{3.8.13d}$$

where

$$\omega_j = \sqrt{\frac{k_j^*}{m_j^*}} \qquad 2\zeta_j\omega_j = \frac{c_j^*}{m_j^*} \qquad \beta_j = \frac{\sum_{i=1}^n m_i\phi_{ij}}{\sum_{i=1}^n m_i\phi_{ij}^2} \tag{3.8.14}$$

Each of the above equations is the equation of motion for a SDOF system for which the natural circular frequency is ω_n, and the damping factor is ζ_n. In addition, the input motion is scaled by the participation factor β_i.

From the equation $\{x\} = [\phi]\{x^*\}$, the solution $\{x\}$ will be evaluated with $[\phi]$ and $\{x^*\}$.

$$\{x\} = [\phi]\{x^*\} \tag{3.8.15a}$$

For a "n" degree of freedom system,

$$\begin{aligned} \{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} &= \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{bmatrix} \begin{Bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{Bmatrix} \\ &= \begin{Bmatrix} \phi_{11}x_1^* + \phi_{12}x_2^* + \cdots + \phi_{1n}x_n^* \\ \phi_{21}x_1^* + \phi_{22}x_2^* + \cdots + \phi_{2n}x_n^* \\ \vdots \\ \phi_{n1}x_1^* + \phi_{n2}x_2^* + \cdots + \phi_{nn}x_n^* \end{Bmatrix} \end{aligned} \tag{3.8.15b}$$

Mode Superposition

Calculate the response of each mode, solving the equation of motion of a SDOF system, e.g. Eq.(3.8.13d) by any appropriate method, e.g. by Duhamel integral.

$$x_j^* = \frac{1}{m_j^*\,\omega_{\mathrm{d}j}} \int_0^t p_j^*(\tau) e^{-\zeta_j\omega_j(t-\tau)} \sin\omega_{\mathrm{d}j}(t-\tau)d\tau \tag{3.8.16a}$$

$$x_j^* = -\frac{1}{\omega_{\mathrm{d}j}} \int_0^t \beta_j\ddot{x}_{\mathrm{g}}(\tau)\, e^{-\zeta_j\omega_j(t-\tau)} \sin\omega_{\mathrm{d}j}(t-\tau)d\tau \tag{3.8.16b}$$

Where x_j^* is obtained as a time history response for the input ground motion $\beta_j\ddot{x}_{\mathrm{g}}$. Summing up the responses of all modes, we can determine the total response of the system using Eq.(3.8.15a).

Modal Analysis

Determine the maximum of x_j^* for each mode using a response spectrum, i.e. $x_{j\,\max}^*$ is given by $\beta_j S_{\mathrm{d}}(\omega_j)$, where $S_{\mathrm{d}}(\omega_j)$ is the ordinate of the displacement response spectrum.

Then the maximum response of x can be estimated, e.g. using square root of sum of the squares (SRSS) as

$$x_{i\max} \approx \sqrt{\sum_{j=1}^{n}\{\beta_j\,\phi_{ij}\,S_{\mathrm{d}}(\omega_j)\}^2} \tag{3.8.17a}$$

The maximum response of x for a "n" DOF system is

$$\{x_{\max}\} = \begin{Bmatrix} x_{1\max} \\ x_{2\max} \\ \vdots \\ x_{n\max} \end{Bmatrix} \approx \begin{Bmatrix} \sqrt{\left\{\beta_1\phi_{11}S_{\mathrm{d}}(\omega_1)\right\}^2 + \left\{\beta_2\phi_{12}S_{\mathrm{d}}(\omega_2)\right\}^2 + \cdots + \left\{\beta_n\phi_{1n}S_{\mathrm{d}}(\omega_n)\right\}^2} \\ \sqrt{\left\{\beta_1\phi_{21}S_{\mathrm{d}}(\omega_1)\right\}^2 + \left\{\beta_2\phi_{22}S_{\mathrm{d}}(\omega_2)\right\}^2 + \cdots + \left\{\beta_n\phi_{2n}S_{\mathrm{d}}(\omega_n)\right\}^2} \\ \vdots \\ \sqrt{\left\{\beta_1\phi_{n1}S_{\mathrm{d}}(\omega_1)\right\}^2 + \left\{\beta_2\phi_{n2}S_{\mathrm{d}}(\omega_2)\right\}^2 + \cdots + \left\{\beta_n\phi_{nn}S_{\mathrm{d}}(\omega_n)\right\}^2} \end{Bmatrix} \tag{3.8.17b}$$

(2) Mode Superposition

The mode superposition procedure is summarized as follows.

i) Mass matrix and stiffness matrix

Calculate the mass matrix $[m]$ and stiffness matrix $[k]$.

ii) Mode shapes and natural frequencies

Determine the mode shapes $\{\phi\}$ and natural frequencies ω, solving the following equation by any appropriate method.

$$\big([k] - \omega^2[m]\big)\{\phi\} = \{0\} \tag{3.8.18a}$$

or

$$\big|[k] - \omega^2[m]\big| = 0 \tag{3.8.18b}$$

iii) Generalized mass and load or participation factor

Calculate the generalized mass and load for each mode, using the following formulae.

$$m_j^* = \{\phi_j\}^{\mathrm{T}}[m]\{\phi_j\} \tag{3.8.19a}$$

$$p_j^* = \{\phi_j\}^{\mathrm{T}}\{p\} \tag{3.8.19b}$$

Or calculate the participation factor.

$$\beta_j = \frac{\{\phi_j\}^{\mathrm{T}}[m]\{1\}}{\{\phi_j\}^{\mathrm{T}}[m]\{\phi_j\}} \tag{3.8.20}$$

iv) Uncoupled equations of motion

Then the equation of motion for each mode becomes as follows.

$$\ddot{x}_j^* + 2\zeta_j\omega_j\dot{x}_j^* + \omega_j^2 x_j^* = \frac{p_j^*}{m_j^*} \tag{3.8.21a}$$

or

$$\ddot{x}_j^* + 2\zeta_j\omega_j\dot{x}_j^* + {\omega_j}^2 x_j^* = -\beta_j\ddot{x}_{\mathrm{g}} \tag{3.8.21b}$$

v) Modal response

Calculate the response for each mode, solving Eq.(3.8.21) by any appropriate method, e.g. by Duhamel integral or step-by-step integration [see Section 2.4(1)].

vi) Total response

Summing up the response for each mode, we can determine the total response of the structure.

$$\{x\} = [\phi]\{x^*\} \tag{3.8.22}$$

(3) Modal Analysis

It should be noted that the mode superposition procedure gives us not the approximate solution but the exact solution. Whereas, the modal analysis using the SRSS (Square Root of Sum of Squares) method gives the approximate solution in a stochastic manner. This method is very similar to the mode superposition method except that in modal analysis the response of each mode is determined by the response spectrum and the maximum total response is given in stochastic manner, e.g. SRSS. The procedure is summarized as follows.

i) Mass matrix and stiffness matrix

This step is exactly the same as to the mode superposition procedure.

ii) Mode shapes and natural frequencies

This step is also the same as to the mode superposition procedure. It should be noted, however, that there is no need to calculate all mode shapes and corresponding frequencies. Because usually only first few modes have a controlling influence on the response of the system, it is enough to calculate the mode shapes and corresponding natural frequencies for the first few influencial modes.

iii) Participation factors

This step is again similar to the mode superposition procedure. The participation factor β_j for each mode is calculated as follows.

$$\beta_j = \frac{\{\phi_j\}^{\mathrm{T}}[m]\{1\}}{\{\phi_j\}^{\mathrm{T}}[m]\{\phi_j\}} \tag{3.8.23a}$$

or

$$\beta_j = \frac{\sum_{i=1}^{n} m_i\phi_{ij}}{\sum_{i=1}^{n} m_i{\phi_{ij}}^2} \tag{3.8.23b}$$

iv) Maximum response for each mode

Determine the maximum response for each mode which can be given by the product of mode shape, participation factor and spectral value at the corresponding frequency and damping ratio, if any.

$$x_{ij\mathrm{max}} = \beta_j\, \phi_{ij}\, S_{\mathrm{d}}(\omega_j) \tag{3.8.24}$$

In order to determine the velocity or acceleration, the spectal value is obtained from the velocity response spectrum $S_{\mathrm{v}}(\omega_n)$ or acceleration response spectrum $S_{\mathrm{a}}(\omega_n)$.

v) Estimation of maximum response

Estimate the maximum total response of the system from each modal response by stochastic manner, e.g. SRSS.

$$x_{i\mathrm{max}} \approx \sqrt{\textstyle\sum_{j=1}^{n} \{\beta_j\, \phi_{ij}\, S_{\mathrm{d}}(\omega_j)\}^2} \tag{3.8.25}$$

(4) Methods for the Estimation of Maximum Response

Although the SRSS method is commonly used for the dynamic analysis of structures, a number of other methods have been proposed to estimate the maximun response.

[**SRSS**] : Square Root of Sum of Squares

In case the natural frequencies are not close to each other, the SRSS gives good estimate of the maximum response.

$$x_{i\mathrm{maxSRSS}} = \sqrt{\textstyle\sum_{j=1}^{n} \{\beta_j\, \phi_{ij}\, S_{\mathrm{d}}(\omega_j)\}^2} \tag{3.8.26}$$

[**ABSSUM**] : Absolute Sum

Since the SRSS sometimes underestimates the maximum response, ABSSUM has been proposed to give the extreme of the maximum response.

$$x_{i\mathrm{maxABS}} = \sum_{j=1}^{n} \left|\beta_j\, \phi_{ij}\, S_{\mathrm{d}}(\omega_j)\right| \tag{3.8.27}$$

[**Average of SRSS and ABSSUM**]

The ABSSUM gives the extreme of the maximum response and usually overestimates it. This is because the maximum response of each mode does not occur simultaneously. Therefore the average of SRSS and ABSSUM has been proposed.

$$x_{i\mathrm{max}} \approx \frac{1}{2}(x_{i\mathrm{maxSRSS}} + x_{i\mathrm{maxABS}}) \tag{3.8.28}$$

[**CQC**] : Complete Quadratic Combination

Since the SRSS does not give good estimate of the maximum response, especially when the natural frequencies are close to each other, CQC has been proposed. The CQC is derived from the random vibration theory, which takes into account the correlation between natural frequencies.

$$x_{i\mathrm{maxCQC}} = \sqrt{\textstyle\sum_{j=1}^{n} \sum_{k=1}^{n} \{\beta_j\, \phi_{ij}\, S_{\mathrm{d}}(\omega_j)\}\; \rho_{jk}\; \{\beta_k\, \phi_{ik}\, S_{\mathrm{d}}(\omega_k)\}} \tag{3.8.29a}$$

$$\rho_{jk} = \frac{8\sqrt{\zeta_j\,\zeta_k}\,(\zeta_j\, +\, r_{jk}\,\zeta_k)\, r_{jk}^{2/3}}{(1\, -\, r_{jk}^2)^2\, +\, 4\zeta_j\,\zeta_k\, r_{jk}\,(1\, +\, r_{jk}^2) +\, 4(\zeta_j^2 + \zeta_k^2)\, r_{jk}^2} \tag{3.8.29b}$$

where, ζ_j and ζ_k are the damping ratios for the j-th and k-th mode, respectively, and r_{jk} is the ratio of the j-th mode natural frequency to the k-th mode natural frequency.

All modes having significant contribution to total structural response should be considered in the above Eqs.(3.8.29a) and (3.8.29b).

[**Example 3.7**]

Calculate the maximum acceleration response of a two story building as shown in Fig.E3.7a, using SRSS method. The building is subjected to the earthquake excitation whose response spectrum is given in Fig.E3.7b, assuming the maximum ground acceleration is 0.3g, and $W_1 = W_2 = 10\,000$ (kN) and $k_1 = k_2 = 1\,000$ (kN/cm).

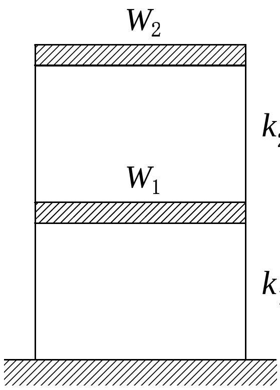


Fig.E3.7a

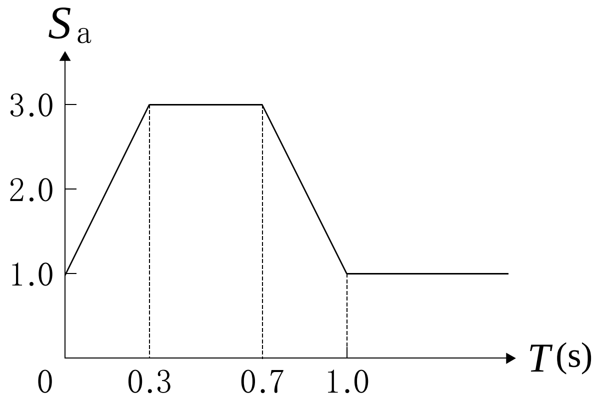


Fig.E3.7b

[**Solution**]

$$m_1 = m_2 = \frac{10\,000}{980} = 10.2\,\text{kN s}^2/\text{cm}$$

The mass matrix is

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 10.2 & 0 \\ 0 & 10.2 \end{bmatrix}$$

The stiffness matrix is

$$[k] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} 2\,000 & -1\,000 \\ -1\,000 & 1\,000 \end{bmatrix}$$

The equation of motion for an undamped system is,

$$[m]\{\ddot{x}\} + [k]\{x\} = 0$$

The mode shapes $\{\phi\}$ are given by

$$\big([k] - \omega^2[m]\big)\{\phi\} = \{0\} \tag{E3.7}$$

Then the frequency equation, $\big|[k] - \omega^2[m]\big| = 0$, becomes

$$\begin{vmatrix} k_{11} - \omega^2 m_1 & k_{12} \\ k_{21} & k_{22} - \omega^2 m_2 \end{vmatrix} = 0$$

$$(k_{11} - \omega^2 m_1)(k_{22} - \omega^2 m_2) - k_{12}k_{21} = 0$$

From the given values of parameters,

$$\begin{aligned} &(2\,000 - 10.2\omega^2)(1\,000 - 10.2\omega^2) - (-1\,000)(-1\,000) = 0 \\ &104\omega^4 - 30\,600\omega^2 + 1\,000\,000 = 0 \\ &1.04\omega^4 - 306\omega^2 + 10\,000 = 0 \end{aligned}$$

Therefore,

$$\begin{aligned} \omega^2 &= \frac{153 \pm \sqrt{153^2 - 10400}}{1.04} = \frac{153 \pm 114}{1.04} = 37.5 \text{ or } 257 \\ \omega_1 &= \sqrt{37.5} = 6.124 \text{ (rad/s)}, \quad T_1 = \frac{2\pi}{\omega_1} = 1.026 \text{ (s)} \\ \omega_2 &= \sqrt{257} = 16.03 \text{ (rad/s)}, \quad T_2 = \frac{2\pi}{\omega_2} = 0.392 \text{ (s)} \end{aligned}$$

From Eq.(E3.7),

$$\begin{bmatrix} k_{11} - \omega^2 m_1 & k_{12} \\ k_{21} & k_{22} - \omega^2 m_2 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = 0$$

The first equation of the above matrix equation becomes

$$(2\,000 - 10.2\omega^2)\phi_1 - 1\,000\phi_2 = 0$$

1) For the first mode, ${\omega_1}^2 = 37.5$

$$1\,618\,\phi_1 - 1\,000\,\phi_2 = 0$$

$$\phi_2 = 1.62\,\phi_1$$

Then, $\{\phi_1\}$ can be chosen arbitrarily as,

$$\begin{Bmatrix} \phi_{11} \\ \phi_{21} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1.62 \end{Bmatrix}$$

2) For the second mode ${\omega_2}^2 = 257$

$$-621\,\phi_1 - 1\,000\,\phi_2 = 0$$

$$\phi_2 = -0.62\,\phi_1$$

Then, $\{\phi_2\}$ can be chosen as,

$$\begin{Bmatrix} \phi_{12} \\ \phi_{22} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -0.62 \end{Bmatrix}$$

The participation factor for the n-th mode is

$$\beta_n = \frac{\sum_{i=1}^{N} m_i \phi_{in}}{\sum_{i=1}^{N} m_i {\phi_{in}}^2}$$

$$\begin{aligned}
\beta_1 &= \frac{m_1\phi_{11} + m_2\phi_{21}}{m_1{\phi_{11}}^2 + m_2{\phi_{21}}^2} = \frac{10.2 \times 1.00 + 10.2 \times 1.62}{10.2 \times 1.00^2 + 10.2 \times 1.62^2} = 0.723 \\
\beta_2 &= \frac{m_1\phi_{12} + m_2\phi_{21}}{m_1{\phi_{12}}^2 + m_2{\phi_{22}}^2} = \frac{10.2 \times 1.00 + 10.2 \times (-0.62)}{10.2 \times 1.00^2 + 10.2 \times (-0.62)^2} = 0.274
\end{aligned}$$

For the fundamental mode $T_1 = 1.026$ (s), the corresponding acceleration response is $S_{\mathrm{a}}(\omega_1) = 1.0 \times 0.3 \times 980 = 294$ (gal). For the second mode $T_2 = 0.392$ (s), the corresponding acceleration response is $S_{\mathrm{a}}(\omega_2) = 3.0 \times 0.3 \times 980 = 882$ (gal).

The maximum acceleration response is computed by SRSS method as,

$$\begin{aligned}
\{\ddot{x}_{\max}\} &\approx \begin{Bmatrix} \sqrt{\{\beta_1\phi_{11}S_{\mathrm{a}}(\omega_1)\}^2 + \{\beta_2\phi_{12}S_{\mathrm{a}}(\omega_2)\}^2} \\ \sqrt{\{\beta_1\phi_{21}S_{\mathrm{a}}(\omega_1)\}^2 + \{\beta_2\phi_{22}S_{\mathrm{a}}(\omega_2)\}^2} \end{Bmatrix} \\
&= \begin{Bmatrix} \sqrt{\{0.723 \times 1.00 \times 294\}^2 + \{0.274 \times 1.00 \times 882\}^2} \\ \sqrt{\{0.723 \times 1.62 \times 294\}^2 + \{0.274 \times (-0.62) \times 882\}^2} \end{Bmatrix} \\
&= \begin{Bmatrix} 322 \\ 375 \end{Bmatrix} \text{(gal)}
\end{aligned}$$

[Example 3.8]

By the modal analysis using SRSS, calculate the maximum responses of a shear type structure as shown in Fig.E3.8a subjected to the earthquake excitation whose velocity response spectrum is given as Fig.E3.8b, where $v_0 = 49$ (cm/s) and $T_{\mathrm{c}} = 1.0$ (s). The weight of each story of the structure from top to bottom is $W_1 = W_2 = W_3 = W = 9\,800$(kN), each story stiffness is $k_1 = 3k$, $k_2 = 5k$ and $k_3 = 6k$, where $k = 200$(kN/cm), and $h_1 = h_2 = h_3 = h = 3$(m).

i) story deflection at each floor.

ii) story shear force at each floor.

iii) overturning moment (OTM) at the base.

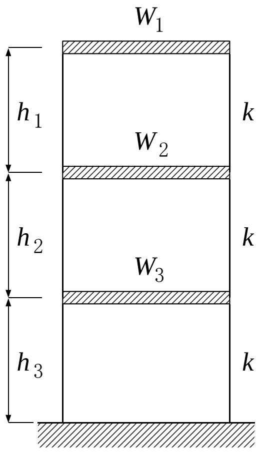


Fig.E3.8a

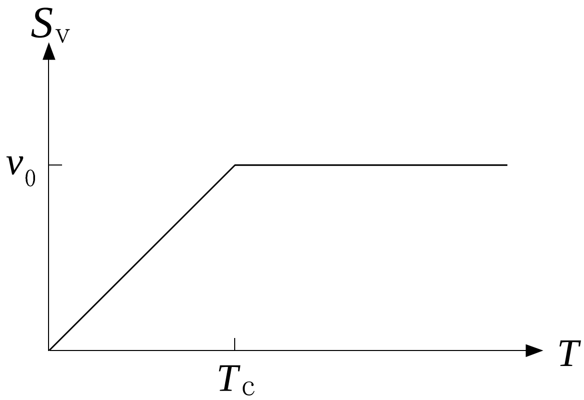


Fig.E3.8b

[**Solution**]

The mass matrix is

$$[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{where } m = \frac{9\,800}{980} = 10$$

The stiffness matrix is

$$[k] = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 + k_3 \end{bmatrix} = k \begin{bmatrix} 3 & -3 & 0 \\ -3 & 8 & -5 \\ 0 & -5 & 11 \end{bmatrix}$$

The mode shapes $\{\phi\}$ are given by

$$\big([k] - \omega^2[m]\big)\{\phi\} = \{0\} \tag{E3.8}$$

Then the frequency equation, $\big|[k] - \omega^2[m]\big| = 0$, becomes

$$\left| k \begin{bmatrix} 3 & -3 & 0 \\ -3 & 8 & -5 \\ 0 & -5 & 11 \end{bmatrix} - m\omega^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$k \begin{vmatrix} 3 - \frac{m}{k}\omega^2 & -3 & 0 \\ -3 & 8 - \frac{m}{k}\omega^2 & -5 \\ 0 & -5 & 11 - \frac{m}{k}\omega^2 \end{vmatrix} = 0$$

Letting $\frac{m}{k}\omega^2 = \lambda$, we have

$$\begin{aligned} (3-\lambda)(8-\lambda)(11-\lambda) - (-3)^2(11-\lambda) - (-5)^2(3-\lambda) &= 0 \\ -\lambda^3 + 22\lambda^2 - 111\lambda + 90 &= 0 \\ (\lambda - 1)(\lambda - 6)(\lambda - 15) &= 0 \end{aligned}$$

Therefore, $\lambda = 1$, 6 or 15.

1) Substituting $\lambda = 1$ to the first and the third equations of Eq.(E3.8), we have

$$\begin{aligned} 2\phi_1 - 3\phi_2 &= 0 \\ -5\phi_2 - 10\phi_3 &= 0 \end{aligned}$$

Then, $\phi_2 = 2\phi_3$, $\phi_1 = 3\phi_3$. Therefore, for the first mode

$$\{\phi_1\} = \begin{Bmatrix} 3 \\ 2 \\ 1 \end{Bmatrix}, \quad \omega_1^2 = \frac{k}{m} = \frac{200}{10} = 20(\mathrm{s}^{-2}), \quad \omega_1 = 4.47(\mathrm{s}^{-1}), \quad T_1 = 1.405(\mathrm{s})$$

2) Substituting $\lambda = 6$, we have

$$\begin{aligned} -3\phi_1 - 3\phi_2 &= 0 \\ -5\phi_2 + 5\phi_3 &= 0 \end{aligned}$$

Then, $\phi_2 = \phi_3$, $\phi_1 = -\phi_3$. Therefore, for the second mode

$$\{\phi_2\} = \begin{Bmatrix} -1 \\ 1 \\ 1 \end{Bmatrix}, \quad \omega_2^2 = 6\frac{k}{m} = 120(\mathrm{s}^{-2}), \quad \omega_2 = 10.96(\mathrm{s}^{-1}), \quad T_2 = 0.574(\mathrm{s})$$

3) Substituting $\lambda = 15$, we have

$$\begin{aligned} -12\phi_1 - 3\phi_2 &= 0 \\ -5\phi_2 - 4\phi_3 &= 0 \end{aligned}$$

Then, $\phi_2 = -0.8\phi_3$, $\phi_1 = 0.2\phi_3$. Therefore, for the third mode

$$\{\phi_3\} = \begin{Bmatrix} 0.2 \\ -0.8 \\ 1 \end{Bmatrix}, \quad \omega_3^2 = 15\frac{k}{m} = 300(\mathrm{s}^{-2}), \quad \omega_3 = 17.32(\mathrm{s}^{-1}), \quad T_3 = 0.363(\mathrm{s})$$

The participation factors are

$$\begin{aligned} \beta_1 &= \frac{10 \times 3 + 10 \times 2 + 10 \times 1}{10 \times 3^2 + 10 \times 2^2 + 10 \times 1^2} = 0.429 \\ \beta_2 &= \frac{10 \times (-1) + 10 \times 1 + 10 \times 1}{10 \times (-1)^2 + 10 \times 1^2 + 10 \times 1^2} = 0.333 \\ \beta_3 &= \frac{10 \times 0.2 + 10 \times (-0.8) + 10 \times 1}{10 \times (0.2)^2 + 10 \times (-0.8)^2 + 10 \times 1^2} = 0.238 \end{aligned}$$

Table E3.8 Calculation sheet for SRSS

	1st mode	2nd mode	3rd mode	max.values
ω_n^2	20	120	300	
ω_n	4.472	10.954	17.321	
T_n	1.405	0.574	0.363	
ϕ_1	3	-1	0.2	
ϕ_2	2	1	-0.8	
ϕ_3	1	1	1	
β_n	0.429	0.333	0.238	
$S_\mathrm{a} = \omega S_\mathrm{v}$	219.0	308.3	308.1	
S_v	49.000	28.126	17.787	
$S_\mathrm{d} = S_\mathrm{v}/\omega_n$	10.96	2.566	1.027	
δ_1(cm)	14.11	-0.85	0.049	14.1
δ_2(cm)	9.40	0.85	-0.196	i) 9.4
δ_3(cm)	4.70	0.85	0.244	4.8
$\ddot{x}_1$(gal)	281.9	-102.7	14.7	300.4
$\ddot{x}_2$(gal)	187.9	102.7	-58.7	222.0
$\ddot{x}_3$(gal)	94.0	102.7	73.3	157.3
p_1(kN)	2 819	$-1\,027$	147	3 004
p_2(kN)	1 879	1 027	-587	2 220
p_3(kN)	940	1 027	733	1 573
V_1(kN)	2 819	$-1\,027$	147	3 004
V_2(kN)	4 698	0	-440	ii) 4 719
V_3(kN)	5 638	1027	293	5 738
OTM (kN· m)	39 465	0	0	iii) 39 465

3.9 Solution by Step-by-step Integration Method

The equation of motion for a MDOF system subjected to the earthquake ground acceleration $\ddot{x}_\mathrm{g}$ is given as follows.

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = -\ddot{x}_\mathrm{g}[m]\{1\} \tag{3.9.1}$$

This equation can be solved directly by a step-by-step integration method as an extension of a SDOF system. The method, i.e. step-by-step integration, is the same as for nonlinear sysytems that will be explained in the next chapter.

The mass matrix and the stiffnes matrix can be evaluated by analyzing the structure. However, the damping matrix can not be evaluated through the analysis of the strucure. Therefore, in this section, it is briefly explained how to construct the damping matrix $[c]$.

Usually we specify a damping property with the damping factor for each mode, e.g. 2% for the first mode, 5% for the second mode, etc.

In order to give a different damping factor for each mode, the damping matrix is constructed as follows. If the orthogonality condition for the damping matrix is satisfied,

$$
\begin{aligned}
[\phi]^{\mathrm{T}}[c][\phi] &= \begin{bmatrix} c_1^* & 0 & \cdots & 0 \\ 0 & c_2^* & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & c_n^* \end{bmatrix} \\
&= \begin{bmatrix} 2\zeta_1\omega_1 m_1^* & 0 & \cdots & 0 \\ 0 & 2\zeta_2\omega_2 m_2^* & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 2\zeta_n\omega_n m_n^* \end{bmatrix} \\
&= \begin{bmatrix} m_1^* & 0 & \cdots & 0 \\ 0 & m_2^* & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & m_n^* \end{bmatrix} \begin{bmatrix} 2\zeta_1\omega_1 & 0 & \cdots & 0 \\ 0 & 2\zeta_2\omega_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 2\zeta_n\omega_n \end{bmatrix} \\
&= \begin{bmatrix} \ddots & & \\ & m_j^* & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \ddots & & \\ & 2\zeta_j\omega_j & \\ & & \ddots \end{bmatrix} \\
&= [m_j^*][2\zeta_j\omega_j]
\end{aligned} \tag{3.9.2}
$$

where, ζ_j is the damping factor for the j-th mode, and ω_j is the natural circular frequency of the j-th mode.

Then, the damping matrix can be evaluated as follows.

$$
\begin{aligned}
[c] &= ([\phi]^{\mathrm{T}})^{-1}[\phi]^{\mathrm{T}}[c][\phi][\phi]^{-1} \\
&= ([\phi]^{\mathrm{T}})^{-1} \begin{bmatrix} m_1^* & 0 & \cdots & 0 \\ 0 & m_2^* & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & m_n^* \end{bmatrix} \begin{bmatrix} 2\zeta_1\omega_1 & 0 & \cdots & 0 \\ 0 & 2\zeta_2\omega_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 2\zeta_n\omega_n \end{bmatrix} [\phi]^{-1} \\
&= ([\phi]^{\mathrm{T}})^{-1} \begin{bmatrix} \ddots & & \\ & m_n^* & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \ddots & & \\ & 2\zeta_n\omega_n & \\ & & \ddots \end{bmatrix} [\phi]^{-1} \\
&= ([\phi]^{\mathrm{T}})^{-1}[m_n^*][2\zeta_n\omega_n][\phi]^{-1}
\end{aligned} \tag{3.9.3}
$$

One of the classical damping matrices is the Rayleigh damping, for which the damping matrix is given as follows.

$$[c] = a_0[m] + a_1[k] \tag{3.9.4}$$

where a_0 and a_1 are the coefficients to be determined from the damping ratios of two different modes. When the damping ratios for the m-th mode and n-th mode are given, these coefficients are given, solving the equation

$$\frac{1}{2}\begin{bmatrix} 1/\omega_m & \omega_m \\ 1/\omega_n & \omega_n \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{Bmatrix} \zeta_m \\ \zeta_n \end{Bmatrix} \tag{3.9.5}$$

When only the damping ratio for the fundamental mode is given, sometimes the damping matrix is estimated as follows.

$$[c] = \frac{2\zeta_1}{\omega_1}[k] \tag{3.9.6}$$

This expression is simple and convenient, but it should be noted that it overestimates the damping ratios for higher modes. This is because Eq.(3.9.6) indicates that the damping ratio is proportional to the frequency.

Chapter 4

Nonlinear Analysis

4.1 Outline of Nonlinear Analysis

In order to analyze a linear structure subjected to arbitrary dynamic loadings, the Duhamel integral is probably the most convenient technique. However, this method and mode superposition can only be applied to linear systems. On the other hand, the response of a structure subjected to severe earthquake motions may exceed the linear range of the materials. Therefore, it is necessary to develop another method that is suitable for nonlinear analysis. The most powerful technique for this purpose is the step-by-step integration. In this method, the response is evaluated for a series of short time increments Δt. The condition of dynamic equilibrium is established at the beginning and end of each time increment, and the nonlinearity of the structure is accounted for by calculating new properties at the beginning of each time increment.

4.2 Nonlinear Response of SDOF Systems

At any instant of time t, the following relation must be satisfied.

$$p_{\mathrm{I}}(t) + p_{\mathrm{d}}(t) + p_{\mathrm{s}}(t) = p(t) \tag{4.2.1}$$

where $p_{\mathrm{I}}(t)$, $p_{\mathrm{d}}(t)$, $p_{\mathrm{s}}(t)$ and $p(t)$ denote the inertia force, damping force, spring (restoring) force and external force, respectively. A short time Δt later, the above equation becomes,

$$p_{\mathrm{I}}(t+\Delta t) + p_{\mathrm{d}}(t+\Delta t) + p_{\mathrm{s}}(t+\Delta t) = p(t+\Delta t) \tag{4.2.2}$$

Subtracting Eq.(4.2.1) from Eq.(4.2.2) yields the incremental form of the equation of motion.

$$\Delta p_{\mathrm{I}}(t) + \Delta p_{\mathrm{d}}(t) + \Delta p_{\mathrm{s}}(t) = \Delta p(t) \tag{4.2.3}$$

where the incremental forces are,

$$\Delta p_{\mathrm{I}}(t) = p_{\mathrm{I}}(t+\Delta t) - p_{\mathrm{I}}(t) = m\Delta\ddot{x}(t) \tag{4.2.4a}$$

$$\Delta p_{\mathrm{d}}(t) = p_{\mathrm{d}}(t+\Delta t) - p_{\mathrm{d}}(t) = c(t)\Delta\dot{x}(t) \tag{4.2.4b}$$

$$\Delta p_{\mathrm{s}}(t) = p_{\mathrm{s}}(t+\Delta t) - p_{\mathrm{s}}(t) = k(t)\Delta x(t) \tag{4.2.4c}$$

$$\Delta p(t) = p(t+\Delta t) - p(t) \tag{4.2.4d}$$

Substituting Eqs.(4.2.4a)∼(4.2.4d) into Eq.(4.2.3) leads to

$$m\Delta\ddot{x}(t) + c(t)\Delta\dot{x}(t) + k(t)\Delta x(t) = \Delta p(t) \tag{4.2.5}$$

Many procedures are available for the numerical integration of the above equation. If we use the linear acceleration method, which is simple but which has been found to yield excellent results, the following relationships are given.

$$\Delta\dot{x}(t) = \ddot{x}(t)\Delta t + \Delta\ddot{x}(t)\frac{\Delta t}{2} \tag{4.2.6a}$$

$$\Delta x(t) = \dot{x}(t)\Delta t + \ddot{x}(t)\frac{\Delta t^2}{2} + \Delta\ddot{x}(t)\frac{\Delta t^2}{6} \tag{4.2.6b}$$

These equations are solved for the incremental acceleration and velocity. Then,

$$\Delta\ddot{x}(t) = \frac{6}{\Delta t^2}\Delta x(t) - \frac{6}{\Delta t}\dot{x}(t) - 3\ddot{x}(t) \tag{4.2.7a}$$

$$\Delta\dot{x}(t) = \frac{3}{\Delta t}\Delta x(t) - 3\dot{x}(t) - \frac{\Delta t}{2}\ddot{x}(t) \tag{4.2.7b}$$

Substituting Eqs.(4.2.7a) and (4.2.7b) into Eq.(4.2.5) leads to

$$m\Big[\frac{6}{\Delta t^2}\Delta x(t) - \frac{6}{\Delta t}\dot{x}(t) - 3\ddot{x}(t)\Big] + c(t)\Big[\frac{3}{\Delta t}\Delta x(t) - 3\dot{x}(t) - \frac{\Delta t}{2}\ddot{x}(t)\Big] + k(t)\Delta x(t) = \Delta p(t) \tag{4.2.8}$$

Rearranging the above equation yields

$$\tilde{k}(t)\Delta x(t) = \Delta\tilde{p}(t) \qquad \text{or} \qquad \Delta x(t) = \frac{\Delta\tilde{p}(t)}{\tilde{k}(t)} \tag{4.2.9}$$

where

$$\tilde{k}(t) = k(t) + \frac{6}{\Delta t^2}m + \frac{3}{\Delta t}c(t) \tag{4.2.10a}$$

$$\Delta\tilde{p}(t) = \Delta p(t) + m\Big[\frac{6}{\Delta t}\dot{x}(t) + 3\ddot{x}(t)\Big] + c(t)\Big[3\dot{x}(t) + \frac{\Delta t}{2}\ddot{x}(t)\Big] \tag{4.2.10b}$$

Therefore, after solving the displacement increment by Eq.(4.2.9), the incremental velocity can be calculated by Eq.(4.2.7b). The initial conditions for the next time step can be given by the addition of the these incremental values to the values at the beginning of the time step. The acceleration should be determined by Eq.(4.2.1) at each time step, in order to minimize the accumulation of any errors which may occur at each step of numerical integration. This is done by using the following equation.

$$\ddot{x}(t) = \frac{1}{m}[\,p(t) - p_{\mathrm{d}}(t) - p_{\mathrm{s}}(t)] \tag{4.2.11}$$

The procedure is summarized as follows.

i) Determine the acceleration of the system under the given initial values and initial conditions by Eq.(4.2.11).

ii) Calculate $\tilde{k}(t)$ and $\Delta\tilde{p}(t)$ by Eqs. (4.2.10a) and (4.2.10b) and determine $\Delta x(t)$ by Eq.(4.2.9).

iii) Calculate the incremental velocity by Eq.(4.2.7b).

iv) Calculate the displacement and the velocity, adding the initial values and the increments.

v) Determine the acceleration by Eq.(4.2.11).

Repeat steps i) to v) until the end of calculation time.

4.3 Nonlinear Response of MDOF Systems

The procedure of nonlinear analysis for MDOF systems is similar to that for SDOF systems. The equilibrium of force increments are given by

$$\{\Delta p_{\mathrm{I}}(t)\} + \{\Delta p_{\mathrm{d}}(t)\} + \{\Delta p_{\mathrm{s}}(t)\} = \{\Delta p(t)\} \tag{4.3.1}$$

The force increments in this equation are

$$\{\Delta p_{\mathrm{I}}(t)\} = \{p_{\mathrm{I}}(t+\Delta t)\} - \{p_{\mathrm{I}}(t)\} = [m]\{\Delta\ddot{x}(t)\} \tag{4.3.2a}$$
$$\{\Delta p_{\mathrm{d}}(t)\} = \{p_{\mathrm{d}}(t+\Delta t)\} - \{p_{\mathrm{d}}(t)\} = [c(t)]\{\Delta\dot{x}(t)\} \tag{4.3.2b}$$
$$\{\Delta p_{\mathrm{s}}(t)\} = \{p_{\mathrm{s}}(t+\Delta t)\} - \{p_{\mathrm{s}}(t)\} = [k(t)]\{\Delta x(t)\} \tag{4.3.2c}$$
$$\{\Delta p(t)\} = \{p(t+\Delta t)\} - \{p(t)\} \tag{4.3.2d}$$

when Eqs. (4.3.2a)$\sim$(4.3.2d) are substituted into Eq.(4.3.1), the incremental equation of motion becomes

$$[m]\{\Delta\ddot{x}(t)\} + [c(t)]\{\Delta\dot{x}(t)\} + [k(t)]\{\Delta x(t)\} = \{\Delta p(t)\} \tag{4.3.3}$$

Adopting the linear acceleration method, we have

$$[\tilde{k}(t)]\{\Delta x(t)\} = \{\Delta\tilde{p}(t)\} \tag{4.3.4}$$

$$[\tilde{k}(t)] = [k(t)] + \frac{6}{\Delta t^2}[m] + \frac{3}{\Delta t}[c(t)] \tag{4.3.5a}$$

$$\begin{aligned}\{\Delta\tilde{p}(t)\} = \{\Delta p(t)\} &+ [m]\Big(\frac{6}{\Delta t}\{\dot{x}(t)\} + 3\{\ddot{x}(t)\}\Big)\\ &+[c(t)]\Big(3\{\dot{x}(t)\} + \frac{\Delta t}{2}\{\ddot{x}(t)\}\Big)\end{aligned} \tag{4.3.5b}$$

When the displacement increment $\{\Delta x(t)\}$ has been determined, the velocity increment is

$$\{\Delta\dot{x}(t)\} = \frac{3}{\Delta t}\{\Delta x(t)\} - 3\{\dot{x}(t)\} - \frac{\Delta t}{2}\{\ddot{x}(t)\} \tag{4.3.6}$$

The displacement and velocity vectors at the end of the time increment are

$$\{x(t+\Delta t)\} = \{x(t)\} + \{\Delta x(t)\} \tag{4.3.7a}$$
$$\{\dot{x}(t+\Delta t)\} = \{\dot{x}(t)\} + \{\Delta \dot{x}(t)\} \tag{4.3.7b}$$

The acceleration vector is given by

$$\{\ddot{x}(t+\Delta t)\} = [m]^{-1}(\{p(t+\Delta t)\} - \{p_d(t+\Delta t)\} - \{p_s(t+\Delta t)\} \tag{4.3.8}$$

Therefore, if we repeat the calculations for Eqs.(4.3.4) through (4.3.8), the response of the MDOF system can be estimated.