Τριγωνομετρικές Σειρές Fourier

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \quad , \; n=0,1\ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \quad , \; n=0,1\ldots$$

Ημιτονικές Σειρές

(Περιττές Συναρτήσεις)

$f(x) = -f(-x)$, $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$, $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}$

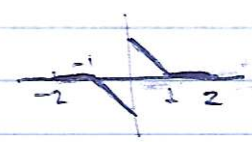
Συνημιτονικές Σειρές

$f(x) = f(-x)$, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$, $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$

ΘΕΩΡΗΜΑ ΣΥΓΚΛΙΣΗΣ.

$$f(x) = \begin{cases} 1-x & , \; 0 \le x \le 1 \\ 0 & , \; 1 \le x \le 2 \end{cases}$$

Ημιτονικό Ανάπτυγμα:



Συνημιτονικό Ανάπτυγμα:

Θ. Parseval

Αν $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$, τότε $\frac{2}{L}\int_0^L [f(x)]^2\,dx = \sum_{n=1}^{\infty} b_n^2$

$$\left(\int_0^L [f(x)]^2 = \sum_{n=1}^{\infty} b_n \int_0^L \sin\frac{n\pi x}{L}\,dx \right)$$

Ονοματολογία: Να αναγνωρίζω Γραμμική – Σχεδόν Γραμμική – Ημιγραμμική

πχ Γραμμική α' τάξης $x \in \mathbb{R}^3$, $\underset{\sim}{x} = (x_1, x_2, x_3)$

$$a_1(\underset{\sim}{x}) u_{x_1} + a_2(\underset{\sim}{x}) u_{x_2} + a_3(\underset{\sim}{x}) u_{x_3} + a_4(x) u = 0$$

Σχεδόν Γραμμική α' τάξης (γραμ μόνο ως προς τους μεγιστοβάθμιους)

$$a_1(\underset{\sim}{x}, u) u_{x_1} + a_2(\underset{\sim}{x}, u) u_{x_2} + a_3(\underset{\sim}{x}, u) u_{x_3} + a_4(\underset{\sim}{x}, u)$$

Ημιγραμ (α' τάξης) (Σχεδόν γραμμ. αλλά οι συντ. συνάρτηση μόνο του $\underset{\sim}{x}$)

$$a_1(x) u_{x_1} + a_2(\underset{\sim}{x}) u_{x_2} + a_3(\underset{\sim}{x}) u_{x_3} + a_4(\underset{\sim}{x}, u) = 0$$

2ης τάξης ~~γραμμική~~ σχεδόν γραμμική $\underset{\sim}{x} \in \mathbb{R}^3$

$$a_1(\underset{\sim}{x}, u, u_{x_1}, u_{x_2}, u_{x_3}) u_{x_1x_1} + a_2(\underset{\sim}{x}, u, \dots) u_{x_2x_2} + a_3(\underset{\sim}{x}, \dots) u_{x_3x_3}$$
$$+ a_4(\underset{\sim}{x}, \dots) u_{x_1x_2} + a_5(\underset{\sim}{x}, \dots) u_{x_1x_3} + a_6(\underset{\sim}{x}, \dots) u_{x_2x_3} +$$
$$+ a_7(\underset{\sim}{x}, \dots) = 0$$

2ης τάξης ημιγραμμική

$$a_1(\underset{\sim}{x}) u_{x_1x_1} + a_2(\underset{\sim}{x}) u_{x_2x_2} + \dots + a_7(\underset{\sim}{x}, u, u_{x_1}, u_{x_2}, u_{x_3}) = 0$$

Παράδειγμα $u_{x_1x_2}(x_1, x_2, x_3) = 0 \rightarrow$ Υπόθεση $\frac{\partial}{\partial x_2}\left[u_{x_1}(x_1, x_2, x_3)\right] = 0$

Ταξινόμηση ημιγραμ. 2ης τάξης.

$$x \in \mathbb{R}^2 \quad a_1(x_1, x_2) u_{x_1x_1} + 2b(x_1, x_2) u_{x_1x_2} + c(x_1, x_2) u_{x_2x_2} + d(\underset{\sim}{x}, u, u_{x_1}, u_{x_2}) = 0$$

$|A - \lambda I| = 0 \Rightarrow A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

1) λ_1, λ_2 ομόσημες → ελλειπτικού τύπου
2) λ_1, λ_2 ετερόσημες → υπερβολικού
3) λ_1, λ_2 μηδενική → παραβολικού

ή $\Delta = b^2 - 4ac$

1) $\Delta < 0$ ελλειπτικού
2) $\Delta > 0$ υπερβολικού
3) $\Delta = 0$ παραβολικού

Παράδειγμα: $x_1^2 u_{x_1x_1} + u_{x_2x_2} + 8x_1x_2 + 25u = 0$

$\Delta = 4 - x_1^2$ ..

$x \in \mathbb{R}^n$ (θ, a, μ)

a → αρνητικές ιδιοτιμές

μ → μηδενικές ιδιοτιμές

θ → θετικές ιδιοτιμές

$(n,0,0)$ ή $(0,n,0)$ → ελλειπτικός

$(n-1,1,0)$ ή $(1,n-1,0)$ → υπερβολικός

$(n-1,0,1)$ ή $(0,n-1,1)$ → παραβολικός

Π.χ. $x_1^2 u_{x_1x_1} + 5u_{x_2x_2} + (1-x_1^2)u_{x_3x_3} + 20u_{x_1} + 8u_{x_2} = 0$

$$\mathbb{A} = \begin{pmatrix} x_1^2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & (1-x_1^2) \end{pmatrix}$$

$\boxed{\text{ΟΧΙ Sturm-Liouville}}$

$\Delta u(x_1,x_2) = 12$, $0<x_1<2$, $0<x_2<3$, $u(0,x_2) = u(2,x_2) = u(x_1,0) = u(x_1,3) = 0$

$u = w+v$, $\Delta w = 0$, $\Delta v = 12$ (μη ομογενής)

$\Delta v = v_{x_1x_1} + v_{x_2x_2}$

$v = a_1x_1^2 + a_2x_2^2 + a_3x_1x_2 + b_1x_1 + b_2x_2 + b_0$

$\Delta v = v_{x_1x_1} + v_{x_2x_2} = 2(a_1+a_2) = 12 \Rightarrow \boxed{a_1 + a_2 = 6}$

$\Delta w = 0$

$w(0,x_2) = -v(0,x_2)$

$w(2,x_2) = -v(2,x_2)$

$w(x_1,0) = -v(x_1,0)$

$w(x_1,3) = -v(x_1,3)$

αν πάρω $v = 6x_1(x_1-2)$ ή $v = 6x_2(x_2-3)$

$w = (0,x_2) = -v(0,x_2) = 0$
$w = (2,x_2) = -v(2,x_2) = 0$

Επομένως έχουμε το πρόβλημα: $\Delta w(x_1,x_2) = 0$

$w(0,x_2) = 0$ $\quad w(x_1,0) = -6x_1(x_1-4)$

$w(2,x_2) = 0$ $\quad w(x_1,3) = -6x_1(x_1-2)$

$$w(x_1,x_2) = X_1(x_1)\cdot X_2(x_2) \Rightarrow \frac{X_1''(x_1)}{X_1(x_1)} = -\frac{X_2''(x_2)}{X_2(x_2)} = k$$

$\frac{X_1''(x_1)}{X_1(x_1)} = k, \quad X_1(0) = X_1(2) = 0$ πρόβλημα Sturm-Liouville

1) $k > 0$
2) $k = 0$
3) $k < 0$

$X_1(x_1) = \sin\frac{n\pi}{2}x_1$

$$k = -\left(\frac{n\pi}{2}\right)^2 \quad \frac{X_2''(x_1)}{X_2(x_2)} = -k = \left(\frac{n\pi}{2}\right)^2 \Rightarrow X_2(x_2) = \begin{cases} \sinh\frac{n\pi}{2}x_2 \\ \cosh\frac{n\pi}{2}x_2 \end{cases}$$

$$\Rightarrow w(x_1,x_2) = \sum_{n=1}^{\infty} \sin\frac{n\pi}{2}x_1\left(a_n\cosh\frac{n\pi}{2}x_2 + b_n\sinh\frac{n\pi}{2}x_2\right)$$

$$w(x_1,0) = \sum_{n=1}^{\infty} \sin\frac{n\pi x_1}{2}, \quad a_n = -6x_1(x_1-2) = \sum_{n=1}^{\infty} f_n \sin\frac{n\pi}{2}x_1$$

όπου $f_n = \frac{2}{2}\int_0^2 [-6x_1(x_1-2)] \sin\frac{n\pi}{2}x_1\, dx_1$

$\Rightarrow$ λόγω ορθ. $a_n = f_n$

Πρόβλημα.

$\Delta u = f(x_1, x_2) = 12$ $\quad u(0, x_2) = u(2, x_2) = u(x_1, 0) = 0$

$u(x_1, 3) = 5x_1^2$

{ μπορώ να διαλέξω μόνο M $V = 6x_1(x_1 - 2) = 0$ }

Πρόβλημα

$u_{xx} + u_{yy} = u_{tt}$

$\Delta u(x, y, t) = u_{tt}(x, y, t)$

$u(x, y, 0) = 0, \quad u_t(x, y, 0) = 5 \sin \frac{5\pi x}{2} \sin \pi y$

$u(0, y, t) = u(2, y, t) = u(x, 0, t) = u(x, 5, t) = 0$

είναι: $\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \frac{T''(t)}{T(t)}$

$\frac{X''(x)}{X(x)} = k, \quad X(0) = X(2) = 0 \quad \to \quad k = -\left(\frac{n\pi}{2}\right)^2, \quad X(x) = \sin \frac{n\pi x}{2}$

$\frac{Y''(y)}{Y(y)} = \lambda, \quad Y(0) = Y(5) = 0 \quad \to \quad \lambda = -\left(\frac{m\pi}{5}\right)^2, \quad Y(y) = \sin \frac{m\pi y}{5}$

$\Rightarrow \frac{T''(t)}{T(t)} = -\left[\left(\frac{n\pi}{2}\right)^2 + \left(\frac{m\pi}{5}\right)^2\right] = -\gamma_{nm}^2 \Rightarrow T(t) = \begin{cases} \cos \gamma_{nm} t \\ \sin \gamma_{nm} t \end{cases}$

$\Rightarrow u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sin \frac{n\pi}{2} x \sin \frac{m\pi}{5} y \cdot \{a_{nm} \cos \gamma_{nm} + b_{nm} \cdot \sin \gamma_{nm}\}$

$u(x, y, 0) = 0 \Rightarrow a_{nm} = 0$

$u_t(x, y, 0) = 5 \sin \frac{5\pi x}{2} \sin \overset{(5)}{\pi} y = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{nm} \cdot \gamma_{nm} \cdot \sin \frac{n\pi}{2} x \sin \frac{m\pi}{5} y$

$\Rightarrow$ λόγω ορθογωνιότητας $b_{nm} = 0 \quad \forall (n, m) \neq (5, 5)$

$b_{55} \gamma_{55} = 5$

Προβλήματα

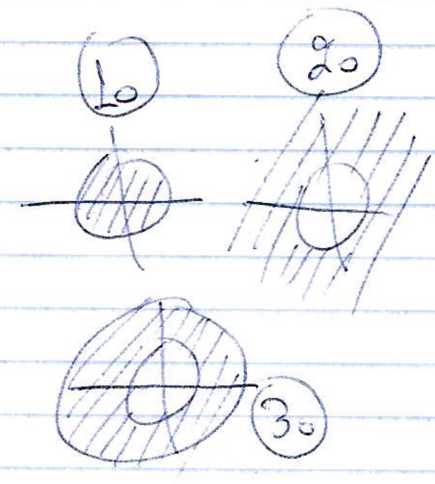
$$\Delta u(\rho,\varphi) = u_{\rho\rho} + \frac{1}{\rho} u_\rho + \frac{1}{\rho^2} u_{\varphi\varphi} = 0 \quad , \quad u(\rho,\varphi) = P(\rho)\cdot\Phi(\varphi)$$

$$\frac{\rho^2 P''(\rho) + \rho P'(\rho)}{P(\rho)} = -\frac{\Phi''(\varphi)}{\Phi(\varphi)} = k$$

Απαιτώ 2π περιοδικότητα της λύσης ως προς $\Phi(\varphi)$

1) $k>0 \quad k=\lambda^2 \Rightarrow \Phi(\varphi) = \begin{cases} \cos\lambda\varphi \\ \sin\lambda\varphi \end{cases} \Big| \ \lambda = 1,2\ldots$

2) $k=0 \Rightarrow \Phi(\varphi) = \begin{cases} 1 \\ \cancel{\varphi} \end{cases}$



3) $k<0 \Rightarrow k=-\lambda^2, \ \Phi(\varphi) = \begin{cases} e^{\lambda\varphi} \\ e^{-\lambda\varphi} \end{cases}$

	$\Phi(\varphi)$	$P(\rho)$	1ο	2ο	3ο
$k=n^2=1,\ldots$	$\cos n\varphi$, $\sin n\varphi$	ρ^n, ρ^{-n}	ρ^n	ρ^{-n}	ρ^n, ρ^{-n}
$k=0$	1	1, $\ln\rho$	1	1	1, $\ln\rho$

$k=n^2$

$$\frac{\rho^2 P''(\rho) + \rho P'(\rho)}{P(\rho)} = n^2 \Rightarrow \rho^2 P''(\rho) + \rho P'(\rho) - n^2 P(\rho) = 0$$

Εξίσωση Euler $\quad \rho = e^t \xrightarrow[\text{αλλαγ.}]{k} \frac{d^2P}{dt^2} = -n^2 P = 0 \Rightarrow P(t) = \begin{cases} e^{nt} \\ e^{-nt} \end{cases}$

① $u(\rho,\varphi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \rho^n (a_n \cos n\varphi + b_n \sin n\varphi)$

② $u(\rho,\varphi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \rho^{-n} (a_n \cos n\varphi + b_n \sin n\varphi)$

③ $u(\rho,\varphi) = a_0 + \ln\rho + \sum_{n=1}^{\infty} \left[(a_n\rho^n + b_n \rho^{-n})\cos n\varphi + (c_n\rho^n + d_n\rho^{-n})\sin n\varphi \right]$

Πρόβλημα

SOS.

$\Delta u(\rho,\varphi) = 2\rho^2 \sin\varphi \quad , \quad 2<\rho<5 \quad , \quad \varphi\in[0,2\pi)$

$\left.\frac{\partial u}{\partial \rho}\right|_{\rho=2} = A+\varphi \quad , \quad \left.\frac{\partial u}{\partial \rho}\right|_{\rho=5} = \cos\varphi$

SOS.

Συνθήκη συμβατότητας

$\int_\Omega \Delta u(\underset{\sim}{x})\, d\underset{\sim}{x} = \int_\Omega \nabla\cdot\nabla u(\underset{\sim}{x})\, d\underset{\sim}{x} = \int_{\partial\Omega} \hat{n}\cdot\nabla u\, ds = \int_{\partial\Omega} \frac{\partial u}{\partial n}\, ds$

$\int_2^5 \int_0^{2\pi} \Delta u(\rho,\varphi)\rho\, d\rho\, d\varphi = \int_0^{2\pi} \cos\varphi\, 5\, d\varphi - \int_0^{2\pi} (A+\varphi)2\, d\varphi \Rightarrow A= \ldots$

$\Delta u(\rho,\varphi) = 2\rho^2 \sin\varphi$

$\left.\frac{\partial u}{\partial \rho}\right|_{\rho=2} = 0 \quad , \quad \left.\frac{\partial u}{\partial \rho}\right|_{\rho=5} = \cos\varphi$

$\boxed{u = w + v}$ (μη ομογενής)

$\Delta w = 0 \quad , \quad \Delta v = A\rho^4 \sin\varphi$

$8A\rho^2 \sin\varphi + 4A\rho^2\sin\varphi - A\rho^2\sin\varphi \rightarrow A = \frac{2}{11} \Rightarrow v = \frac{2}{11}\rho^4 \sin\varphi$

$\Delta w(\rho,\varphi) = 0 \quad , \quad \frac{\partial w}{\partial \rho} = \frac{\partial v}{\partial \rho}$

Fourier

$$a^2 \Delta u(x,y,t) = u_t \qquad (x,y) \in (-\infty,+\infty) \times (-\infty,\infty), \quad t>0$$
$$u(x,y,0) = 8e^{-(x^2+y^2/4)}$$

$$\mathcal{F}\{u_{xx}\} + \mathcal{F}\{u_{yy}\} = \frac{1}{a^2}\mathcal{F}\{u_t\} \Leftrightarrow$$

$$\frac{1}{(\sqrt{2\pi})^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u_{xx} \cdot e^{i(\lambda x+\mu y)}\,dx\,dy + \frac{1}{(\sqrt{2\pi})^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u_{yy}\, e^{i(\lambda x+\mu y)}\,dx\,dy =$$

$$= \frac{1}{a^2}\frac{1}{(\sqrt{2\pi})^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u_t\, e^{i(\lambda x+\mu y)}\,dx\,dy$$

$$(-i\lambda)^2\hat{u}(\lambda,\mu,t) + (-i\mu)^2\hat{u}(\lambda,\mu,t) = \frac{1}{a^2}\frac{\partial}{\partial t}\hat{u}(\lambda,\mu,t)$$

$$\hat{u}(\lambda,\mu,t) = C(\lambda,\mu)e^{-(\lambda^2+\mu^2)t} = \hat{u}(\lambda,\mu,0)e^{-(\lambda^2+\mu^2)t}$$

$$\Rightarrow u(x,y,t) = \frac{1}{(\sqrt{2\pi})^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\hat{u}(\lambda,\mu,t)e^{-i(\lambda x+\mu y)}\,d\lambda\,d\mu$$

ΑΣΚΗΣΗ

$$F_c\{u_{xx}\} = -\lambda^2\hat{u}_c(\lambda,t) - \sqrt{\frac{2}{\pi}}\,u_x(0,t)$$

$$F_s\{u_{xx}\} = -\lambda^2\hat{u}_s(\lambda,t) + \sqrt{\frac{2}{\pi}}\,\lambda u(0,t)$$

$$\boxed{\begin{array}{l} u_t - u_{xx} = 0 \\ u(x,0) = 0 \\ u_x(0,t) = A(t) \end{array}}, \quad x \in (0,+\infty)$$

$$\Rightarrow \frac{\partial \hat{u}_c}{\partial t} + \lambda^2\hat{u}_c(\lambda,t) = -\sqrt{\frac{2}{\pi}}A(t)$$

Γραμ. α' τάξης μη ομογενής.

$\downarrow$

$$\hat{u}_c(\lambda,t) = -\sqrt{\frac{2}{\pi}}\int_0^t A(\tau)e^{-\lambda^2(t-\tau)}$$

u στο μηδέν $\Rightarrow$ ημιτονικός

u' στο μηδέν $\rightarrow$ συνημιτονικός