Κανονική 2013-2014

Θέμα 1ο

$$\underset{\sim}{A} = \begin{bmatrix} 1 & 4 & 0.5 \\ 4 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad , \quad \underset{\sim}{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Μετασχηματίζω το σύστημα έτσι ώστε ο πίνακας να παρουσιάζει διαγώνιο υπεροχή (κατά γραμμή)

$$\underset{\sim}{A}_{επ} = \left[\begin{array}{ccc|c} 1 & 4 & 0.5 & 2 \\ 4 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{array}\right]$$

Εναλλάσσω την 1η με τη 2η γραμμή:

$$\left[\begin{array}{ccc|c} 4 & 1 & 1 & 1 \\ 1 & 4 & 0.5 & 2 \\ 0 & 1 & 2 & 3 \end{array}\right] \qquad \left.\begin{array}{l} |a_{11}| = |4| = 4 \ , \ |a_{12}| + |a_{13}| = |1| + |1| = 2 \\ |a_{22}| = |4| = 4 \ , \ |a_{21}| + |a_{23}| = |1| + |0.5| = 1.5 \\ |a_{33}| = |2| = 2 \ , \ |a_{31}| + |a_{32}| = |0| + |1| = 1 \end{array}\right\} \text{Διαγώνια υπεροχή (κατά γραμμή)}$$

Οπότε
$$\left.\begin{array}{l} 4x_{n+1} = y_n + z_n + 1 \\ 4y_{n+1} = x_n + \frac{z_n}{2} + 2 \\ 2z_{n+1} = z_n + 3 \end{array}\right\} \Rightarrow \begin{array}{l} x_{n+1} = \frac{y_n}{4} + \frac{z_n}{4} + \frac{1}{4} \\ y_{n+1} = \frac{x_n}{4} + \frac{z_n}{8} + \frac{1}{2} \\ z_{n+1} = \frac{z_n}{2} + \frac{3}{2} \end{array}$$

Έστω $x_0 = 0$

$x_1 = 1/4$

$y_1 = 1/2$

$z_1 = 3/2$

$x_2 = \frac{1}{4}\cdot\frac{1}{2} + \frac{3}{2}\cdot\frac{1}{4} + \frac{1}{4} = \frac{1}{8} + \frac{3}{8} + \frac{2}{8} = \frac{6}{8} = \frac{3}{4}$

$y_2 = \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{8}\cdot\frac{3}{2} + \frac{1}{2} = \frac{1}{16} + \frac{3}{16} + \frac{8}{16} = \frac{12}{16} = \frac{6}{8} = \frac{3}{4}$

$z_2 = \frac{3}{4} + \frac{3}{2} = \frac{9}{4}$

$x_3 = \frac{1}{4}\cdot\frac{3}{4} + \frac{1}{4}\cdot\frac{9}{4} + \frac{1}{4} = \frac{3}{16} + \frac{9}{16} + \frac{4}{16} = 1$

$y_3 = \frac{3}{4}\cdot\frac{1}{4} + \frac{1}{8}\cdot\frac{9}{4} + \frac{1}{2} = \frac{6}{32} + \frac{9}{32} + \frac{16}{32} = \frac{31}{32}$

$z_3 = \frac{1}{2}\cdot\frac{9}{4} + \frac{3}{2} = \frac{9}{8} + \frac{12}{8} = \frac{21}{8}$

$$\|\underset{\sim}{x}_n - \bar{x}\|_\infty \leq \frac{\|C_J\|_\infty}{1 - \|C_J\|_\infty} \cdot \|x_n - x_{n-1}\|_\infty$$

Εκτίμηση Σφάλματος

$$\underset{\sim}{x}_3 - \underset{\sim}{x}_2 = \begin{bmatrix} 1 \\ 31/32 \\ 21/8 \end{bmatrix} - \begin{bmatrix} 3/4 \\ 1/4 \\ 9/4 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 7/32 \\ 3/4 \end{bmatrix}$$

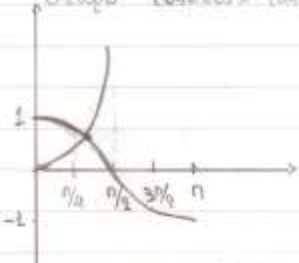
$$\|x_3 - x_2\|_\infty = \frac{3}{4}$$

$$C_J = Q_J^{-1}\cdot P_J = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -0.5 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1/4 & -1/4 \\ -1/4 & 0 & -0.125 \\ 0 & -1/2 & 0 \end{bmatrix} \quad |C_J| = 0.5$$

Οπότε $|\underset{\sim}{x}_3 - \bar{x}|_\infty \leq \frac{0.5}{0.5}\cdot\frac{3}{4} = 0.75$

Θέμα 2°:

Θεωρώ $f(x)=\cos x-\tan x$, $f'(x)=-\sin x-\frac{1}{\cos^2 x}$



Newton Raphson: $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$

$x_0=\frac{\pi}{4}=0,78539$ $\quad x_1=0,78539-\frac{\sqrt{2}/2-1}{-\sqrt{2}/2-\frac{1}{(\sqrt{2}/2)^2}}=0,78539-\frac{0,29289}{2,70711}$

$\Rightarrow x_1=0,67720$

$x_2=0,67720-\frac{(-0,02471)}{-2,27309}=0,66633$

$x_3=0,66633-\frac{(-2,01599\cdot 10^{-4})}{-2,23637}=0,66633-9,01615\cdot 10^{-5}=0,66624$

Θέμα 3ο

Τα 3 σημεία παρεμβολής είναι: $x_0=0$, $x_1=0,2$, $x_2=0,5$

$\therefore f(x_0)=f(0)=1$

$f(x_1)=0,951$

$f(x_2)=0,882$

$$\ell_0=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}=\frac{(x-0,2)(x-0,5)}{(0-0,2)(0-0,5)}=10(x-0,2)(x-0,5)$$

$$\ell_1=\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}=\frac{(x-0)(x-0,5)}{0,2\cdot(-0,3)}=\frac{x\cdot(x-0,5)}{-0,06}$$

$$\ell_2=\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}=\frac{(x-0)(x-0,2)}{0,5\cdot 0,3}=\frac{x(x-0,2)}{0,15}$$

$$P_2(x)=f(x_0)\cdot\ell_0+f(x_1)\cdot\ell_1+f(x_2)\cdot\ell_2=10(x-0,2)(x-0,5)+0,951\cdot\frac{x(x-0,5)}{-0,06}+0,882\cdot\frac{x(x-0,2)}{0,15}$$

$$f(x)-P_2(x)=\frac{f'''(\xi)}{3!}(x-0)(x-0,2)(x-0,5)$$

$$f'(x)=-\frac{1}{4}e^{-x/4}$$

$$f''(x)=\frac{1}{16}e^{-x/4}$$

$$f'''(x)=-\frac{1}{64}e^{-x/4}\Rightarrow|f'''(\xi)|\leq\frac{1}{64}$$

$$g(x)=(x-0)(x-0,2)(x-0,5)=(x^2-0,2x)(x-0,5)=x^3-\frac{x^2}{2}-\frac{x^2}{5}+0,1x=x^3-\frac{7x^2}{10}+\frac{x}{10}$$

$$g'(x)=3x^2-\frac{7x}{5}+\frac{1}{10}\qquad g'(x)=0\Rightarrow 3x^2-1,4x+0,1=0\quad \Delta=1,4^2-4\cdot 3\cdot 0,1=0,76$$

$$x_{1,2}=\frac{1,4\pm 0,87}{6}\Rightarrow x_1=0,378,\ x_2=0,088$$

$$g''(x)=6x-\frac{7}{5}$$

$g''(0,378)=0,868$ τοπ. ελ. $g''(0,088)=-0,868$ τοπ. μεγ.

$g(0,378)=0,378^3-\frac{7\cdot 0,378^2}{10}+\frac{0,378}{10}=-8,909\cdot 10^{-3}\Rightarrow$ μέγιστο κατά απόλ. τιμή

$g(0,088)=0,088^3-\frac{7\cdot 0,088^2}{10}+\frac{0,088}{10}=4,061\cdot 10^{-3}$

Οπότε $|f(x)-P_2(x)|\leq\frac{1}{64\cdot 6}\cdot 8,909\cdot 10^{-3}=2,138\cdot 10^{-5}$

Θέμα 4ο:

$I = \int_0^1 \frac{1}{1+x} dx$, $n=6$ $\qquad f(x) = \frac{1}{1+x}$

$h = \frac{1-0}{6} = 0{,}167$ $\qquad x_0 = 0,\ x_1 = 0{,}167,\ x_2 = 0{,}333,\ x_3 = 0{,}5,\ x_4 = 0{,}667,\ x_5 = 0{,}834,\ x_6 = 1$

$I = h \cdot (f_0 + 4f_1 + 4f_2 + 4f_3 + 4f_4 + 4f_5 + f_6)$

$f_0 = f(x_0) = 1$
$f_1 = f(x_1) = 0{,}857$
$f_2 = f(x_2) = 0{,}75$
$f_3 = f(x_3) = 0{,}667$
$f_4 = f(x_4) = 0{,}6$
$f_5 = f(x_5) = 0{,}545$
$f_6 = f(x_6) = 0{,}5$

Οπότε $I = 0{,}167\,(1 + 4 \cdot 0{,}857 + 4 \cdot 0{,}75 + 4 \cdot 0{,}667 + 4 \cdot 0{,}6 + 4 \cdot 0{,}545 + 0{,}5) = 2{,}534$

$e = \frac{b-a}{180} \cdot h^4 \cdot f''''(n) = \frac{1}{180} \cdot 0{,}167^4 \cdot f''''(n) = 4{,}2870 \cdot 10^{-6} \cdot f''''(n)$

$f'(x) = -\frac{1}{(1+x)^2}$, $\quad f''(x) = \frac{2}{(1+x)^3}$, $\quad f'''(x) = \frac{-6}{(1+x)^4}$, $\quad f''''(x) = \frac{24}{(1+x)^5}$

Η $f''''(x)$ μεγιστοποιείται για $x = 0$: $f''''(n) = 24$

Οπότε $e = 1{,}0288 \cdot 10^{-4}$